College of Engineering
Bio. Eng. Dept.
Subject: Numerical Analysis
Third Stage

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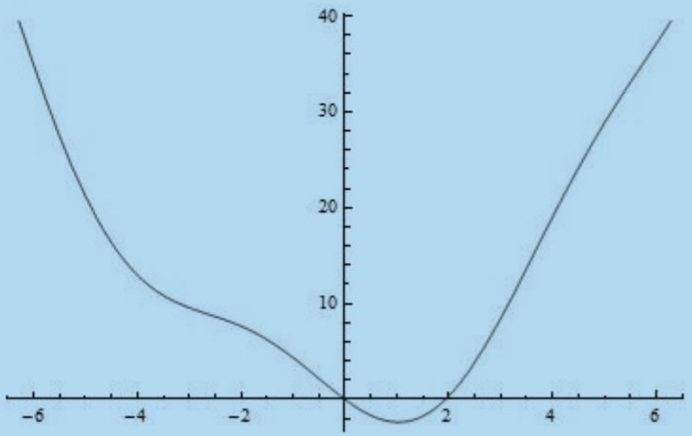
Lecture 1

Introduction

Numerical methods are mathematical techniques used for solving mathematical problems that cannot be solved or are diffcult to solve (example: eq.1). The numerical solution is an approximate numerical value for the solution. Although numerical solutions are an approximation, they can be very accurate.

Example: Find the roots of the following equation

$$f(x) = x^2 - 4\sin(x) = 0$$



In many numerical methods, the calculations are executed in an iterative manner until a desired accuracy is achieved.

Introduction

❖ Today, numerical methods are used in fast electronic digital computers that make it possible to execute many tedious and repetitive calculations that produce accurate (even though not exact) solutions in a very short time.

❖ For every type of mathematical problem there are several numerical techniques that can be used. The techniques differ in accuracy, length of calculations, and diffculty in programming.

Example: to solve a nonlinear equation of the form $f(x) = 0$ one can use among others

- Bisection method
- The Newton-Raphson method
- Secant method
- The fixed-point iteration method

Errors in numerical solutions

Since numerical solutions are an approximation, and since the computer program that executes the numerical method might have errors, a numerical solution needs to be examined closely. There are three major sources of error in computation:
human errors, **truncation errors**, and **round-off errors**.

1. Human errors

Typical human errors are arithmetic errors, and/or programming errors: These errors can be very hard to detect unless they give obviously incorrect solution. In discussing errors, we shall assume that human errors are not present.

2. Round-off errors

- Understanding how round off errors occur because digital computers have a limited ability to represent numbers
- Understanding why floating-point numbers have limits on their range and precision.

3. Truncation errors

- Recognizing that truncation errors occur when exact mathematical formulations are represented by approximations.
- Knowing how to use the Taylor series to estimate truncation errors.
- Understanding how to write forward, backward, and centered finite difference approximations of the first and second derivatives.
- Recognizing that efforts to minimize truncation errors can sometimes increase round off errors.

Errors in numerical solutions

Error Definitions

True error (Et): the difference between the true value and the approximation.

$$Et = True\ value - approximation$$

- **Absolute error ($|Et|$):** the absolute difference between the true value and the approximation.
- **True fractional relative error:** the true error divided by the true value.

True fractional relative error = (true value - approximation)/true value

- **Relative error (ε_t):** the true fractional relative error expressed as a percentage.

$$\varepsilon_t = true\ fractional\ relative\ error * 100\%$$

Errors in numerical solutions

Error Definitions

When not knowing true values.

$$\varepsilon_a = \frac{approximation\ error}{approximation} \text{x}100\%$$

but how can we calculate the approximation error without true value?

$$\varepsilon_a = \frac{present\ approximation - previous\ approximation}{present\ approximation} \text{x}100\%$$

for numerical methods based on iteration

Numerical methods for finding the roots of a function

The roots of a function $f_{(x)}$ are defined as the values for which the value of the function becomes equal to zero. So, finding the roots of $f_{(x)}$ means solving the equation

$$f_{(x)} = 0$$

Example 1 : If $f_{(x)} = ax^2 + bx + c$ is a quadratic polynomial, the roots are given by the well-known formula

$$x_1, x_2 = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

Example 2 : For a polynomial of degree 3 or higher, it is sometimes (but not very often!) possible to find the roots by factorising the polynomial

$$f_x = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3) \qquad \textit{so the roots are } 1, 2 \textit{ and } 3$$

$$f_x = x^4 - 16 = (x^2 - 4)(x^2 + 4) \qquad \textit{so the roots are } 2 \textit{ and } -2$$

For a large number of problems it is, however, not possible to find exact values for the roots of the function so we have to find approximations instead.

1. The bisection method

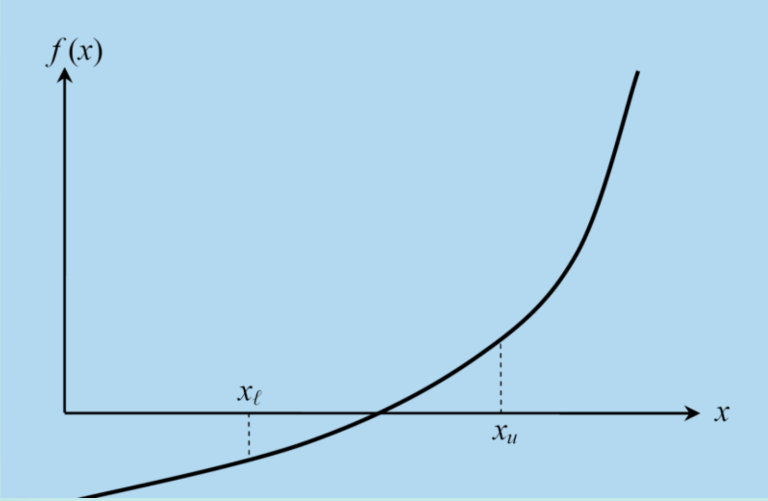
One of the first numerical methods developed to find the root of a nonlinear equation was the bisection method

The method is based on the following theorem:

Theorem

An equation $f_{(x)} = 0$, where is $f_{(x)}$ a real continuous function, has at least one root between $x_{(l)},$ and $x_{(u)}$ if $f_{(xl)} f_{(xu)} < 0$

Note that if $f_{(xl)} f_{(xu)} > 0$, there may or may not be any root between $x_{(l)},$ and $x_{(u)}$. If $f_{(xl)} f_{(xu)} < 0$, then there may be more than one root between $x_{(l)},$ and $x_{(u)}$. So the theorem only guarantees one root between $x_{(l)},$ and $x_{(u)}$



Algorithm for the bisection method

The steps to apply the bisection method to find the root of the equation $f_{(x)} = 0$ are

1. Choose $x_{(l)},$and $x_{(u)}$ as two guesses for the root such that $f_{(xl)} f_{(xu)} < 0$, or in other words, $f_{(x)}$ changes sign between $x_{(l)},$and $x_{(u)}$.

2. Estimate the root, $x_{(m)},$ of the equation $f_{(x)} = 0$ as the mid-point between $x_{(l)},$and $x_{(u)}$

$$x_m = \frac{x_{(l)} + x_{(u)}}{2}$$

3. Now check the following

a) If $f_{(xl)} f_{(xm)} < 0$, then the root lies between $x_{(l)},$and $x_{(m)}$; then $x_{(l)} = x_{(l)}$ and $x_{(u)} = x_{(m)}$

b) If $f_{(xl)} f_{(xm)} > 0$, then the root lies between $x_{(m)},$and $x_{(u)}$; then $x_{(l)} = x_{(m)}$ and $x_{(u)} = x_{(u)}$

c) If $f_{(xl)} f_{(xm)} = 0$; then the root is $x_{(m)}$. Stop the algorithm if this is true.

4. Find the new estimate of the root

$$x_m = \frac{x_{(l)} + x_{(u)}}{2}$$

Find the absolute relative approximate error as

$$|\varepsilon_a| = \left| \frac{x^{new}_{m} - x^{old}_{m}}{x^{new}_{m}} \right| * 100\%$$

where

x^{new}_{m} = estimated root from present iteration

x^{old}_{m} = estimated root from previous iteration

Example: Find the roots of the following equation $x^6 - x - 1 = 0$

accurate to within $\varepsilon = 0.001$.

First we have to find an interval where the function $f_x = x^6 - x - 1$ changes sign. Easy to see that $f_{(1)} f_{(2)} < 0$ so we take $a = 1$ and $b = 2$. Then $c = (a+b)/2 = 1.5$ and the bisection algorithm is detailed in the following table.

n	a	b	c	b-c	f(c)
1	1.0000	2.0000	1.5000	0.5000	8.8906
2	1.000	1.5000	1.2500	0.2500	1.5647
3	1.0000	1.2500	1.1250	0.1250	-0.0977
4	1.1250	1.2500	1.1875	0.0625	0.6167
5	1.1250	1.1875	1.1562	0.0312	0.2333
6	1.1250	1.1562	1.1406	0.0156	0.0616
7	1.1250	1.1406	1.1328	0.0078	-0.0196
8	1.1328	1.1406	1.1367	0.0039	0.0206
9	1.1328	1.1367	1.1348	0.0020	0.0004
10	1.1328	1.1348	1.1338	0.00098	-0.0096

Note:

that after 10 steps we have

$b - c = 0.00098 < 0.001$

hence: the required root approximation is

$c = 1.1338$.