College of Engineering
Bio. Eng. Dept.
Subject: Numerical Analysis
Third Stage

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Lecture 2

2. The Newton-Raphson method

Introduction

Recall that the equation of a straight line is given by the equation

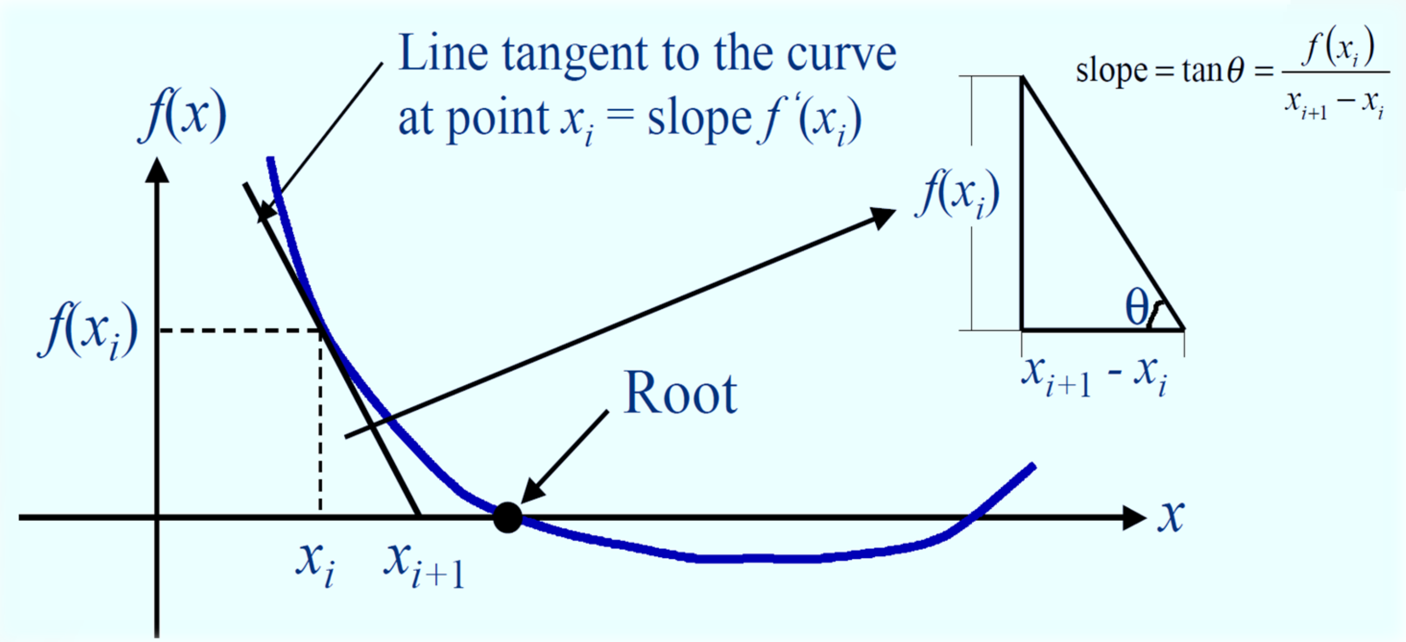
$$y = mx + n \qquad (1)$$

where m is called the slope of the line. (This means that all points (x, y) on the line satisfy the equation above.) . If we know the slope m and one point (x_0, y_0) on the line, equation (1) becomes:

$$y - y_0 = m(x - x_0) \qquad (2)$$

Introduction

Assume we need to find a root of the equation $f\ (x) = 0$. Consider the graph of the function $f\ (x_i)$ and an initial estimate of the root, x_i. To improve this estimate, take the tangent to the graph of $f\ (x)$ through the point $(x_i\ , f(x_i))$ and let x_{i+1} be the point where this line crosses the horizontal axis.



Derivation of Newton-Raphson Method

Graphical Derivation

✓ Newton-Raphson Iteration

From the previous figure,

$$\text{Slope} = -f'(x_i) = \frac{df(x)}{dx}\Big|_{x=x_i} = \frac{f(x_i) - 0}{x_{i+1} - x_i}$$

$$x_{i+1} - x_i = +\frac{f(x_i)}{-f'(x_i)}$$

$$x_{i+1} = x_i + \frac{f(x_i)}{-f'(x_i)}$$

Derivation of Newton-Raphson Method

Note :

that since the root of the function relating $f(x)$ and x is the value of x when $f(x_{i+1}) = 0$ at the intersection, hence,

$$f(x_{i+1}) = 0 = f(x_i) + (x_{i+1} - x_i) f'(x_i)$$

$$(x_{i+1} - x_i) f'(x_i) = -f(x_i)$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

x_i = value of the root at iteration i
x_{i+1} = a revised value of the root at iteration $i+1$
$f(x_i)$ =value of the function at iteration i
$f'(x_i)$ =derivative of $f(x_i)$ evaluated at iteration i

Example:

Use the Newton-Raphson iteration method to estimate the root of the following function employing an initial guess of x_0 =0:

$$f(x) = e^{-x} - x$$

Solution:

Lets find the derivative of the function first

$$f'(x) = \frac{df(x)}{dx} = -e^{-x} - 1$$

The initial guess is x_0 =0, hence,

✓ i=0

$f(0) = e^{0} - 0 = 1$ ≫ $f'(0) = -e^{-0} - 1 = -1 - 1 = -2$

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ ≫ $x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$ ≫ $x_1 = 0 - \frac{1}{-2} = 0.5$

Now x_1 =0.5, hence,

✓ i=1

$f(0.5) = e^{-0.5} - 0.5 = 0.1065$ ≫ $f'(0.5) = -e^{-0.5} - 1 = -1.6065$

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ ≫ $x_{1+1} = x_1 - \frac{f(x_1)}{f'(x_1)}$ ≫ $x_2 = 0.5 - \frac{0.1065}{-1.6065} = 0.5663$

Now x_2 =0.5663, hence,

✓ i=2

$f(0.5663) = e^{-0.5663} - 0.5663 = 0.001322$ ≫ $f'(0.5663) = -e^{-0.5663} - 1 = -1.567622$

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ ≫ $x_{2+1} = x_2 - \frac{f(x_2)}{f'(x_2)}$ ≫ $x_3 = 0.5663 - \frac{0.001322}{-1.567622} = 0.5671$

Now x_3 =0.5671, hence,

✓ i=3

$f(0.5671) = e^{-0.5671} - 0.5671 = 0.00006784$ ≫ $f'(0.5671) = -e^{-0.5671} - 1 = -1.56716784$

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ ≫ $x_{3+1} = x_3 - \frac{f(x_3)}{f'(x_3)}$ ≫ $x_4 = 0.5671 - \frac{0.00006784}{-1.56716784} = 0.5671$

Homework:

Find the roots of the following equation $x^6 - x - 1 = 0$
Using an initial guess of x_0 =1.5 ?