College of Engineering
Mech. Eng. Dept.
Subject: Strength of Materials
Second Class   Lecturer: Kadhim M. Al-sahlani

Course weekly Outline

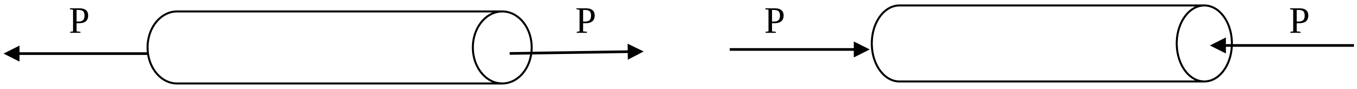
week	Topics Covered
1	Simple stress
2	Shear Stress-Bearing stress
3	Thin walled pressure
4	Simple strain-Hook's law
5	Axial deformations
6	Statically Indeterminate Members
7	Thermal Stresses
8	Torsion
9	=
10	Shear and moment in beams
11	Shear force and bending moment diagrams
12	Stresses in beams
13	=
14	Shear stress in beams
15	Curved beams
16	Slope and deflection in beams
17	Integration method
18	Moment area method
19	Castigliano's method
20	Combined stresses
21	=
22	Moher's circle for stresses
23	=
24	Moher's circle for strain
25	Columns
26	Euler's equation for column
27	Theories of failure
28	=
29	Helical Springs
30	=

Textbook	Hibbler	Mechanics of Materials			
	Singer	Mechanics of Materials			
References	Gere	Mechanics of Materials			
	Hearn	Mechanics of Materials			
Course Assessment	Term Tests	Laboratory	Quizzes	Project	Final Exam
	35	10	5	----	50

***S**trength of materials* is a branch of applied mechanics that deals with the behavior of solid bodies subjected to various types of loading. The solid bodies include axially-loaded bars, shafts, beams, and columns. The objective of analysis will be the determination of the stresses, strains, and deformations produced by the loads.

Simple Stress (σ):

If a cylindrical bar is subjected to a direct ***pull*** or ***push*** along its axis, then it is said to be subjected to ***tension*** or ***compression***.



In SI systems of units load is measured in ***Newton (N)*** or ***Kiloewton (KN)*** or ***Meganewton (MN).***

Normal stress (σ) : is the intensity of normal force per unit area

$$\text{Stress} = \frac{Load}{Area}$$

$$\sigma = \frac{P}{A}$$

stress may thus be ***compressive*** or ***tensile*** depending on the nature of the load and will be measured in units of Newton per square meter ***(N/m²)***. This unit, called ***Pascal***

1 Pa=1 N/m^2
1 KPa=1000 Pa=10^3 Pa
1 MPa=10^6 Pa
1 GPa=10^9 Pa

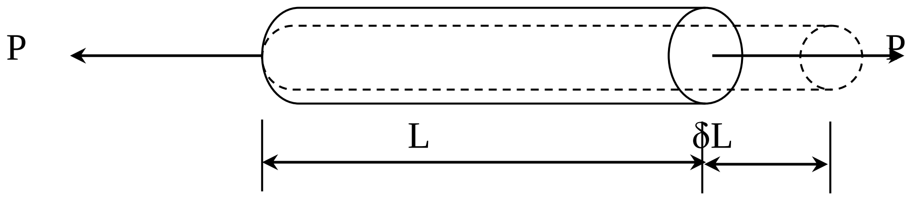
In the U.S. customary or foot-pound-second system of units, express stress in pounds per square inch ***(Psi)*** or kilopound per square inch ***(Ksi)***

Normal Strain (ε):

If a bar is subjected to a direct load, and hence a stress, the bar will change in length. If the bar has an original length (L) and changes in length by an amount (δL), the strain produced is defined as follows

$$\text{Strain}(\varepsilon) = \frac{change\,in\,length}{original\,length}$$

$$\varepsilon = \frac{\delta L}{L}$$



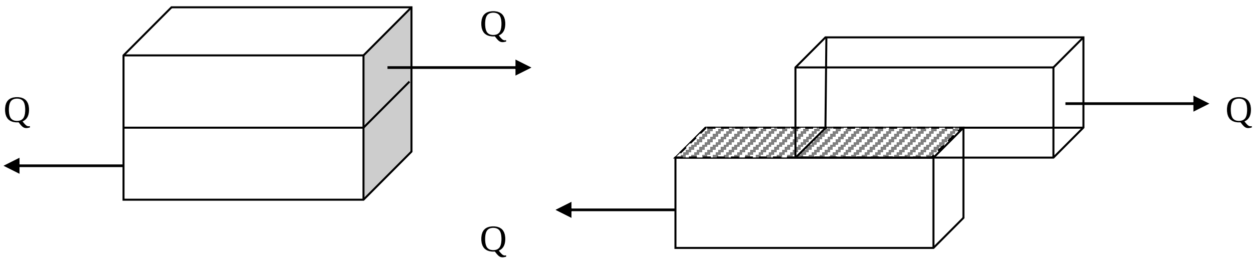
Strain is thus a measure of the deformation of the material and is non-dimensional, i.e. it has no units. Tensile stresses and strains are considered positive sense. Compressive stresses and strains are considered negative in sense.

Shear Stress (τ) and Bearing Stress (σ_b):

Shearing stress differs from both tensile and compressive stress in that it is caused by forces acting along or parallel to the area resisting the forces, whereas tensile and compressive stresses are caused by forces perpendicular to the areas on which they act. For this reason, tensile and compressive stresses are called normal stresses, whereas a shearing stress may be called a tangential stress.

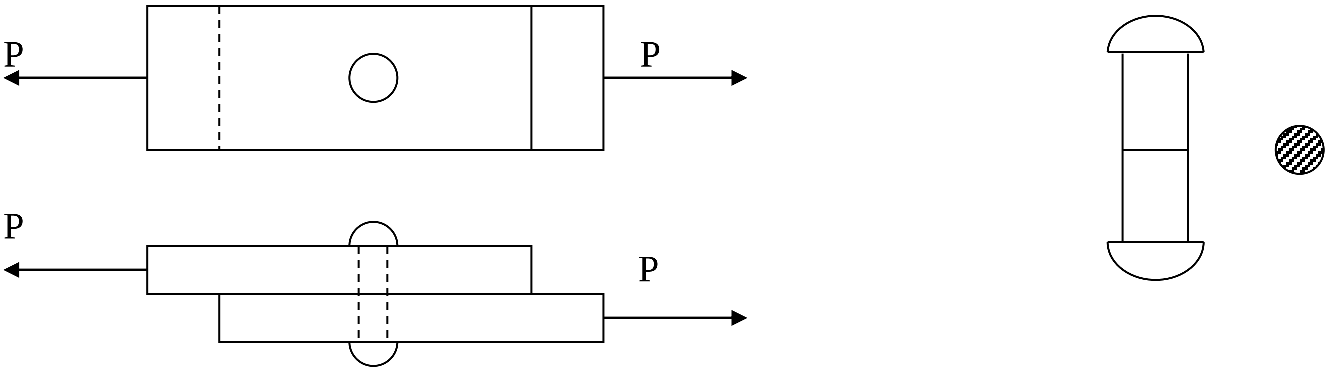
A shearing stress is produced whenever the applied loads cause one section of a body to tend to slide past its adjacent section.

$$\text{Shear stress} = \frac{Shear\,load}{Area\,resisting\,shear}$$



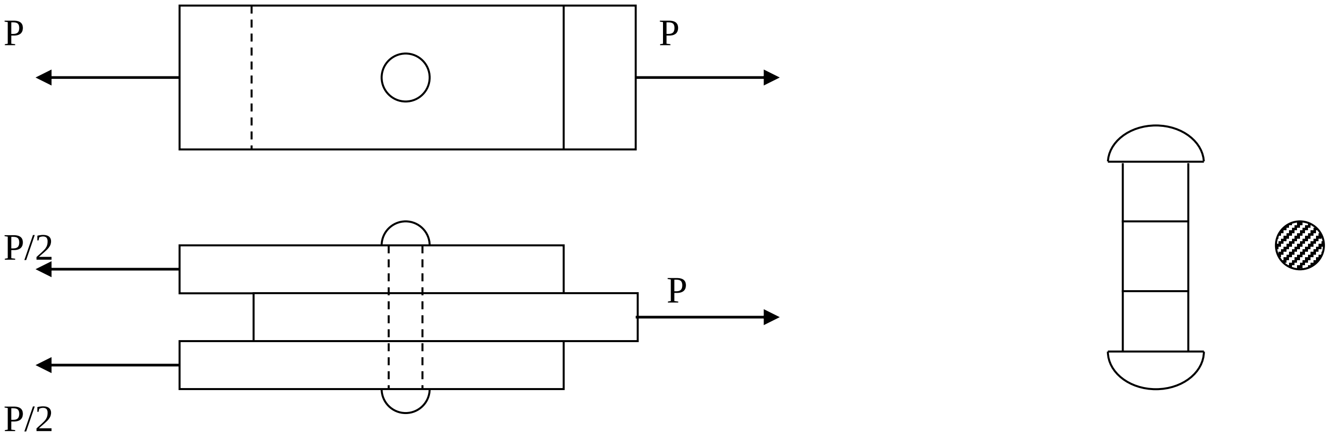
$$\tau = \frac{Q}{A}$$

Area resisting shear is the shaded area as shown above.



Single shear stress

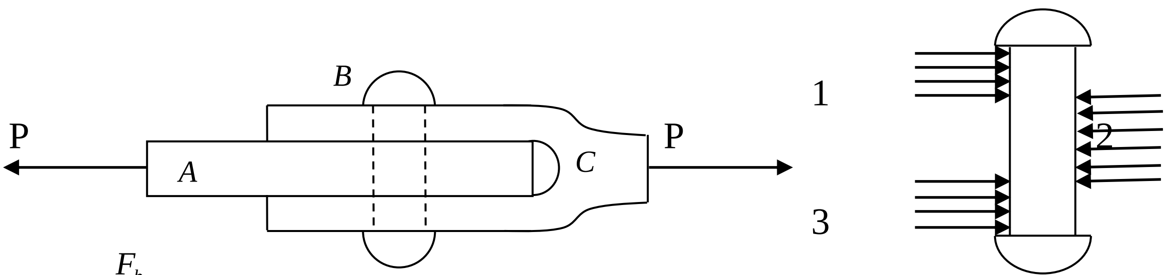
$$\tau = \frac{P}{A}$$



Double shear stress

$$\tau = \frac{P/2}{A}$$

Bearing stress is a normal stress that is produced by the compression of one surface against another. The bearing area is defined as the projected area of the curved bearing surface.

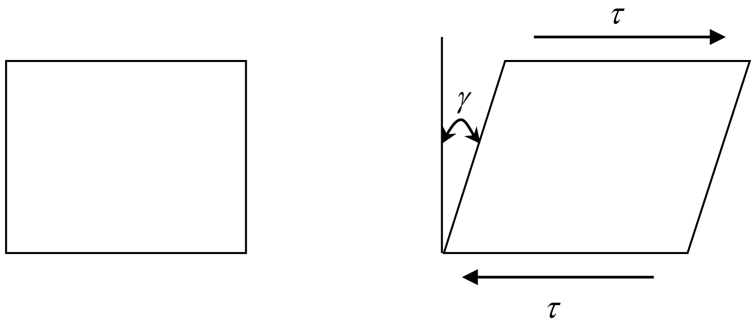


$$\sigma_b = \frac{F_b}{A_b}$$

Consider the bolted connection shown above, this connection consists of a flat bar A, a clevis C, and a bolt B that passes through holes in the bar and clevis. Consider the bearing stresses labeled 1, the projected area A_b on which they act is rectangle having a height equal to the thickness of the clevis and a width equal to the diameter of the bolt, the bearing force F_b represented by the stresses labeled 1 is equal to P/2. The same area and the same force apply to the stresses labeled 3. For the bearing stresses labeled 2, the bearing area A_b is a rectangle with height equal to the thickness of the flat bar and width equal to the bolt diameter. The corresponding bearing force F_b is equal to the load P.

Shear Strain (γ):

Shear strain is a measure of the distortion of the element due to shear. Shear strain is measured in radians and hence is non-dimensional, i.e. it has no units.



Elastic Materials-Hook's Law:

A material is said to be elastic if it returns to its original, when load is removed. In elastic material, stress is proportional to strain. Hook's law therefore states that:

$$\text{Stress } (\sigma) \propto \text{strain } (\varepsilon)$$

$$\frac{stress}{strain} = \text{constant}$$

Within the elastic limit, i.e. within the limits in which Hook's law applies, it has been shown that:

$$\frac{\sigma}{\varepsilon} = E$$

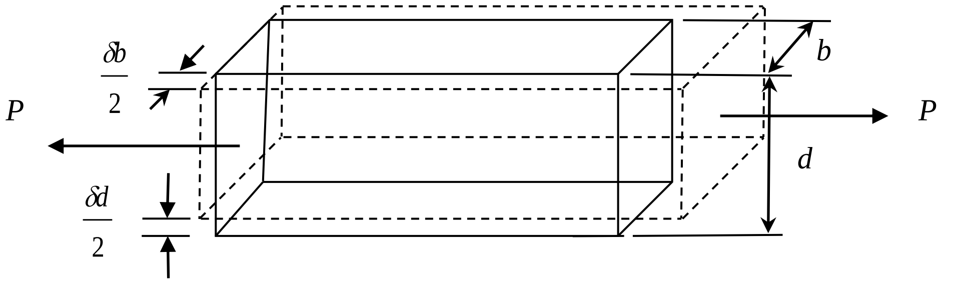
This constant is given the symbol E and termed the modulus of elasticity or Young's modulus.

Poisson's Ratio (υ):

Consider the rectangular bar shown below subjected to a tensile load. Under the action of this load the bar will increase in length by an amount δL giving a longitudinal strain in the bar of:

$$\varepsilon_L = \frac{\delta L}{L}$$

The bar will also exhibit a reduction in dimensions laterally i.e. its breadth and depth will both reduce.



The associated lateral strains will both be equal, will be of opposite sense to the longitudinal strain, and will be given by:

$$\varepsilon_{lat} = -\frac{\delta d}{d} = -\frac{\delta b}{b}$$

Poisson's ratio is the ratio of the lateral and longitudinal strains and always constant

$$\text{Poisson's ratio}=\frac{Lateral\,Strain}{Longitudinal\,Strain}$$

$$\nu = \frac{\delta d / d}{\delta L / L}$$

$$\text{Longitudinal Strain}=\frac{\sigma}{E}$$

$$\text{Lateral Strain}=\upsilon\frac{\sigma}{E}$$

Modulus of Rigidity (G):

For materials within the elastic range the shear strain is proportional to the shear stress producing it.

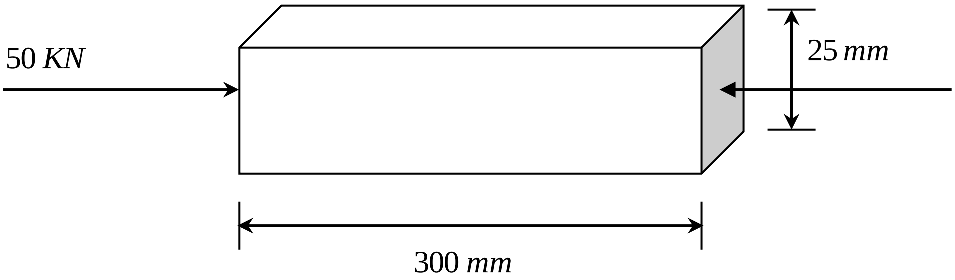
$$\tau \propto \gamma$$

$$\frac{Shear\,Stress}{Shear\,Strain}=\text{Constant}$$

$$\frac{\tau}{\gamma}=\text{G}$$

The constant G is termed the modulus of rigidity.

Example 1: A ***25 mm*** square cross-section bar of length ***300 mm*** carries an axial compressive load of ***50 KN***. Determine the stress set up in the bar and its change of length when the load is applied. For the bar material ***E=200 GN/m²***.



Cross-section area of the bar(A)=$25\times10^{-3}\times25\times10^{-3}=625\times10^{-6}$ m^2

$$\sigma=\frac{P}{A}$$

$$\sigma=\frac{50\times10^{3}}{625\times10^{-6}}=80000000\ \text{N/m}^2$$

$$\sigma=80\ \text{MN/ m}^2$$

$$\varepsilon=\frac{\sigma}{E}$$

$$\varepsilon=\frac{80\times10^{6}}{200\times10^{9}}=0.0004$$

$$\delta L=\varepsilon L$$

$$\delta L=0.0004\times300\times10^{-3}=0.12\times10^{-3}\text{m}$$

$$\delta L=0.12\ \text{mm}$$

Example 2: Two circular bars, one of brass and the other of steel, are to be loaded by a shear load of ***30 KN***. Determine the necessary diameter of the bars a) in single shear b) in double shear, if the shear stress in the two materials must not exceed ***50 MN/m²*** and ***100 MN/m²*** respectively.

a) <u>Single Shear</u>

$$\tau = \frac{F}{A} \Longrightarrow A = \frac{F}{\tau}$$

- ***For brass material***

A= $\frac{30\times10^3}{50\times10^6}$ =0.0006 m^2

A=πr^2 $\Longrightarrow$ $r = \sqrt{\frac{A}{\pi}}$ $\Longrightarrow$ $r = \sqrt{\frac{0.0006}{\pi}}$

r=13.8197×10^{-3} m

the diameter of the bar (d)=27.639×10^{-3} m

- ***For steel material***

A= $\frac{30\times10^3}{100\times10^6}$ =0.0003 m^2

$r = \sqrt{\frac{0.0003}{\pi}}$ =9.772×10^{-3} m

the diameter of the bar (d)=19.544×10^{-3} m

<u>b) Double Shear</u>

$$\tau = \frac{F}{2A} \Longrightarrow A = \frac{F}{2\tau}$$

- ***For brass material***

A= $\frac{30\times10^3}{2\times50\times10^6}$ =0.0003 m^2

$r = \sqrt{\frac{0.0003}{\pi}}$ =9.772×10^{-3} m

the diameter of the bar (d)=19.544×10^{-3} m

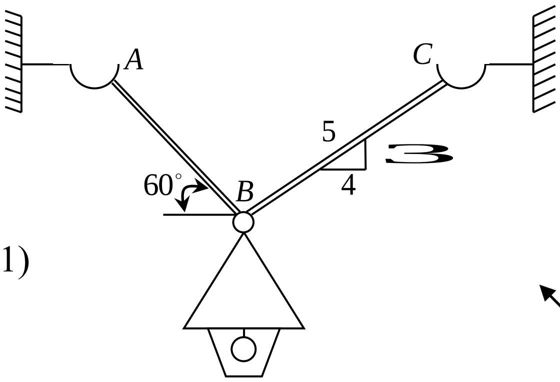
- ***For steel material***

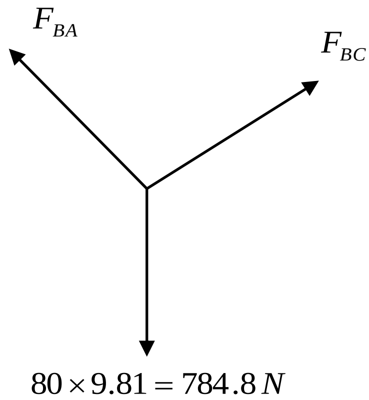
A= $\frac{30\times10^3}{2\times100\times10^6}$ =0.00015 m^2

$r = \sqrt{\frac{0.00015}{\pi}}$ =6.909×10^{-3} m

the diameter of the bar (d)=13.819×10^{-3} m

Example 3: The ***80 kg*** lamp is supported by two rods ***AB*** and ***BC*** as shown. If ***AB*** has a diameter of ***10 mm*** and ***BC*** has a diameter of ***8 mm***, determine the average normal stress in each rod.





$$\sum F_x = 0$$

$$F_{BC} \times \frac{4}{5} - F_{BA} \times \cos 60 = 0$$

$F_{BC}=0.625F_{BA}$ ……………..(1)

$$\sum F_y = 0$$

$$F_{BC} \times \frac{3}{5} + F_{BA} \times \sin 60 - 784.8 = 0$$

$F_{BC}=1308-1.44337F_{BA}$ ……….(2)

$1308-1.44337F_{BA}=0.625F_{BA}$

$F_{BA}=632.38$ N

$F_{BC}=395.2375$ N

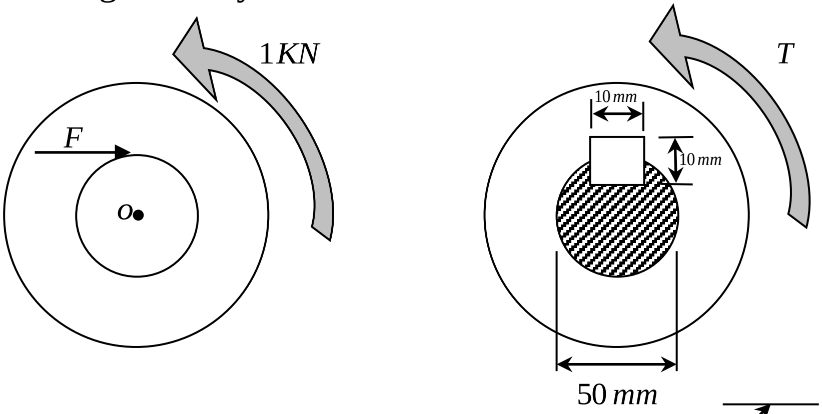
$$\sigma_{BA} = \frac{F_{BA}}{A_{BA}} = \frac{632.38}{\pi(5\times10^{-3})^2}$$

$\sigma_{BA}=8.051877\times10^6$ Pa
$\sigma_{BA}=8.051877$ MPa

$$\sigma_{BC} = \frac{F_{BC}}{A_{BC}} = \frac{395.2375}{\pi(4\times10^{-3})^2}$$

$\sigma_{BC}=7.863149\times10^6$ Pa
$\sigma_{BC}=7.863149$ MPa

Example 4: Shafts and pulleys are usually fastened together by means of a key, as shown. Consider a pulley subjected to a turning moment ***T*** of ***1 KN.m*** keyed by a ***10 mm×10 mm×75 mm*** key to the shaft. The shaft is ***50 mm*** in diameter. Determine the shear stress on a horizontal plane through the key.



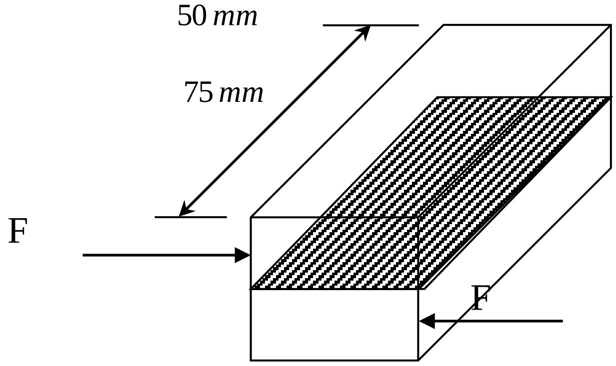
$$\sum M_o = 0$$

$$1\times10^3 - F\times0.025 = 0$$

F=40000 N

F=40 KN

$$\tau = \frac{F}{A}$$



A is the shaded area

$$\tau = \frac{40\times10^3}{10\times10^{-3}\times75\times10^{-3}}$$

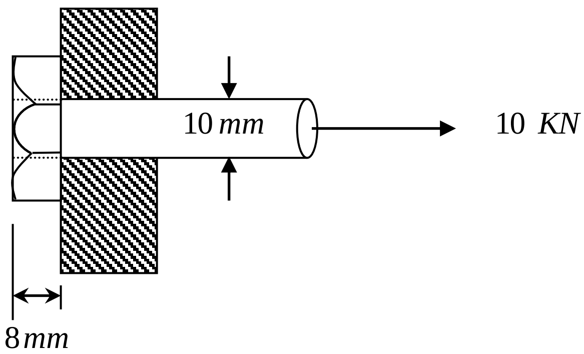
$\tau = 53.333\times10^6$ N/m^2

$\tau = 53.333$ MN/m^2

Example 5: Consider a steel bolt ***10 mm*** in diameter and subjected to an axial tensile load of ***10 KN*** as shown. Determine the average shearing stress in the bolt head, assuming shearing on a cylindrical surface of the same diameter as the bolt.

$A = \pi dt$

$A = \pi\times10\times10^{-3}\times8\times10^{-3} = 0.000251327$ m^2



$$\tau = \frac{F}{A}$$

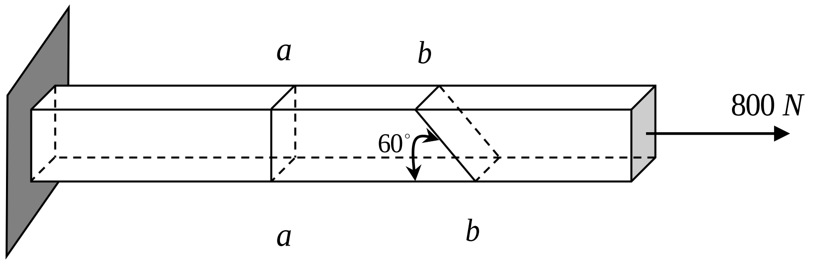
$$\tau = \frac{10\times10^3}{0.000251327}$$

$\tau = 39.7888\times10^6$ N/m^2

$\tau = 39.7888$ MN/m^2

Example 6: The bar shown has a square cross section for which the depth and thickness are ***40 mm***. If an axial force of ***800 N*** is applied along the centroidal axis of the bar's cross sectional area, determine the average normal stress and average shear stress acting on the material along a) section plane ***a-a*** and b) section plane ***b-b***.

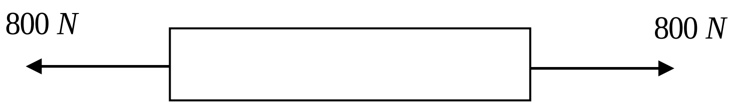
a) section plane ***a-a***



$$\sigma = \frac{P}{A}$$

$$\sigma = \frac{800}{40\times10^{-3}\times40\times10^{-3}}$$

σ=500 KN/m^2

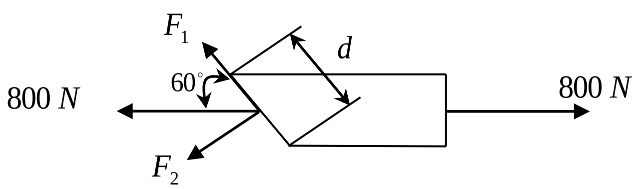


$$\tau = \frac{F}{A}$$

F=0

τ=0

b) section plane ***b-b***



$$d = \frac{40}{\sin 60} = 46.188 \text{ mm}$$

$$\sigma = \frac{F_2}{A}$$

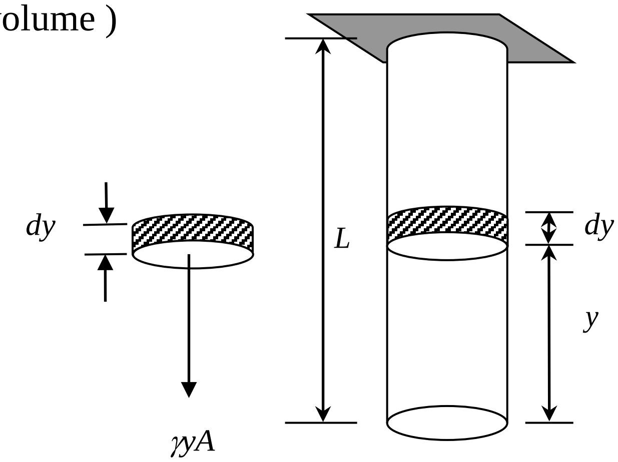
$$\sigma = \frac{800\sin 60}{46.188\times10^{-3}\times40\times10^{-3}} = 375 \text{ KN/m}^2$$

$$\tau = \frac{F_1}{A}$$

$$\tau = \frac{800\cos 60}{46.188\times10^{-3}\times40\times10^{-3}} = 216.50645 \text{ KN/m}^2$$

Example 7: Determine the total increase of length of a bar of constant cross section hanging vertically and subject to its own weight as the only load. The bar is initially straight.

γ: is the specific weight (weight/unit volume)
A: is the cross-sectional area



$$d\delta = \frac{\gamma A y dy}{AE}$$

$$\delta = \int_0^L d\delta$$

$$\delta = \int_0^L \frac{\gamma A y dy}{AE}$$

$$= \frac{\gamma A}{AE} \int_0^L y dy$$

$$= \frac{\gamma A}{AE} \frac{1}{2} y^2 \Big|_0^L$$

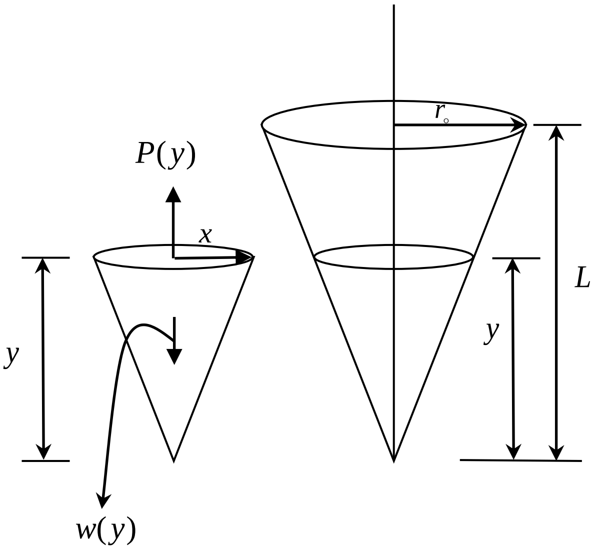
$$\delta = \frac{\gamma A}{2AE} L^2$$

$$\delta = \frac{\gamma AL.L}{2AE}$$

W=γAL

$$\delta = \frac{W.L}{2AE}$$

Example 8: A member is made from a material that has a specific weight γ and modulus of elasticity ***E***. If its formed into a cone having the dimensions shown, determine how far its end is displaced due to gravity when its suspended in the vertical position.



$$\frac{x}{r_\circ} = \frac{y}{L}$$

$$x = r_\circ \frac{y}{L}$$

$$v = \frac{\pi}{3} x^2 y$$

$$w(y) = \gamma v = \gamma \frac{\pi}{3} x^2 y$$

$$= \frac{\pi}{3} \gamma \frac{r_\circ^{\,2} y^2}{L^2} y$$

$$w(y) = \frac{\pi}{3} \frac{\gamma r_\circ^{\,2}}{L^2} y^3$$

From equilibrium P(y)=w(y)

$$P(y) = \frac{\pi}{3} \frac{\gamma r_\circ^{\,2}}{L^2} y^3$$

$$A(y) = \pi x^2 = \pi \frac{r_\circ^{\,2}}{L^2} y^2$$

$$d\delta = \frac{P(y)dy}{A(y)E} = \frac{\frac{\pi}{3} \gamma \frac{r_\circ^{\,2}}{L^2} y^3 dy}{\pi \frac{r_\circ^{\,2}}{L^2} y^2 E}$$

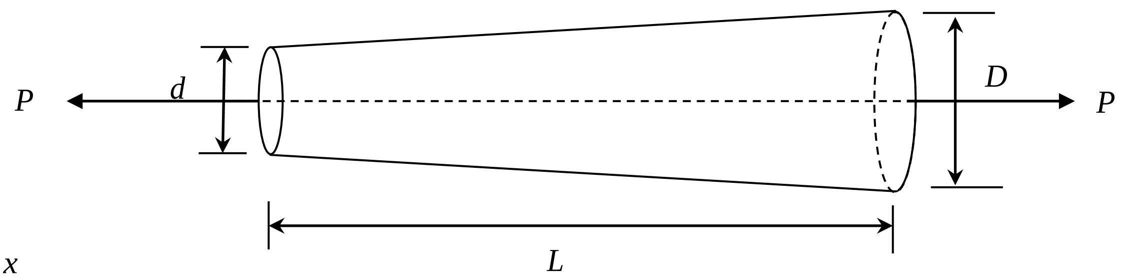
$$d\delta = \frac{\gamma}{3E} y dy$$

$$\delta = \int_0^L d\delta = \int_0^L \frac{\gamma y dy}{3E}$$

$$\delta = \frac{\gamma L^2}{6E}$$

Example 9: A solid truncated conical bar of circular cross section tapers uniformly from a diameter d at its small end to D at the large end. The length of the bar is L. Determine the elongation due to an axial force P applied at each end as shown.

$$r = \frac{d}{2} + y$$

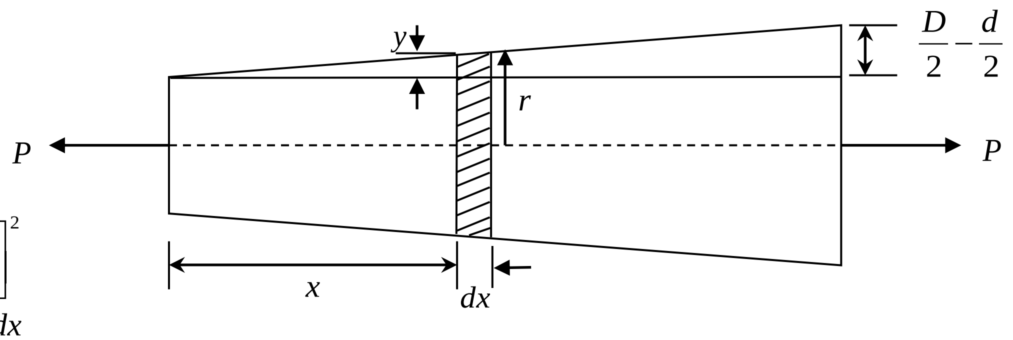


$$\frac{y}{x} = \frac{\frac{D}{2} - \frac{d}{2}}{L}$$

$$y = \left(\frac{D}{2} - \frac{d}{2}\right)\frac{x}{L}$$

$$r = \frac{d}{2} + \left(\frac{D}{2} - \frac{d}{2}\right)\frac{x}{L}$$

A(x)=πr^2



$$A(x) = \pi\left[\frac{d}{2} + \left(\frac{D}{2} - \frac{d}{2}\right)\frac{x}{L}\right]^2$$

$$d\delta = \frac{Pdx}{A(x)E} = \frac{Pdx}{\pi\left[\frac{d}{2} + \left(\frac{D}{2} - \frac{d}{2}\right)\frac{x}{L}\right]^2 E}$$

$$\delta = \int_0^L \frac{Pdx}{\pi\left[\frac{d}{2} + \left(\frac{D}{2} - \frac{d}{2}\right)\frac{x}{L}\right]^2 E}$$

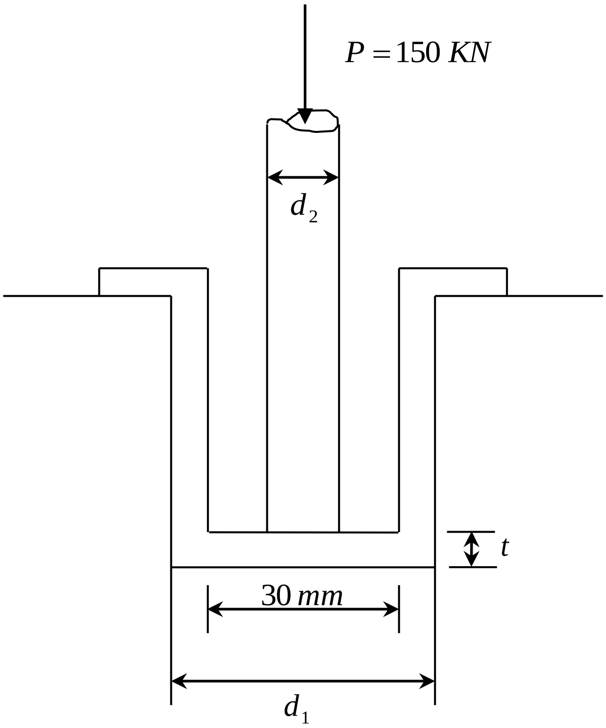
$$= -\frac{P}{E\frac{\pi}{L}\left(\frac{D}{2} - \frac{d}{2}\right)}\left[\frac{d}{2} + \left(\frac{D}{2} - \frac{d}{2}\right)\frac{x}{L}\right]^{-1}\Bigg|_0^L$$

$$= -\frac{PL}{E\pi\left(\frac{D}{2} - \frac{d}{2}\right)\left[\frac{d}{2} + \left(\frac{D}{2} - \frac{d}{2}\right)\frac{x}{L}\right]}\Bigg|_0^L = -\frac{PL}{E\pi\left(\frac{D}{2} - \frac{d}{2}\right)\left[\frac{d}{2} + \frac{D}{2} - \frac{d}{2}\right]} + \frac{PL}{E\pi\left(\frac{D}{2} - \frac{d}{2}\right)\frac{d}{2}}$$

$$= -\frac{PL}{E\pi\left(\frac{D^2}{4} - \frac{dD}{4}\right)} + \frac{PL}{E\pi\left(\frac{Dd}{4} - \frac{d^2}{4}\right)} = \frac{4PL}{E\pi}\left[-\frac{1}{D^2 - dD} + \frac{1}{Dd - d^2}\right]$$

$$\delta = \frac{4PL}{\pi dDE}$$

Example 10: Determine the smallest dimensions of the circular shaft and circular end cop if the load it is required to support is ***150 KN***. The allowable tensile stress, bearing stress, and shear stress is $\boldsymbol{(\sigma_t)_{allow}=175\ MPa}$, is $\boldsymbol{(\sigma_b)_{allow}=275\ MPa}$, and $\boldsymbol{\tau_{allow}=115\ MPa}$.



$(\sigma_b)_{allow}=\frac{F_b}{A_b}$

$275\times10^6=\frac{150\times10^3}{A_b}$

$A_b=0.0005454\ m^2$

$A_b=\frac{\pi}{4}d_2^2$

$d_2=\sqrt{\frac{4A_b}{\pi}}=\sqrt{\frac{4\times0.0005454}{\pi}}$

$d_2=0.026353\ m=26.353\ mm$

$(\sigma_t)_{allow}=\frac{P}{A}$

$175\times10^6=\frac{150\times10^3}{A}$

$A=0.0008571\ m^2$

$A=\frac{\pi}{4}[d_1^2-(30\times10^{-3})^2]=0.0008571$

$d_1=0.04462\ m=44.62\ mm$

$\tau_{allow}=\frac{F}{A}$

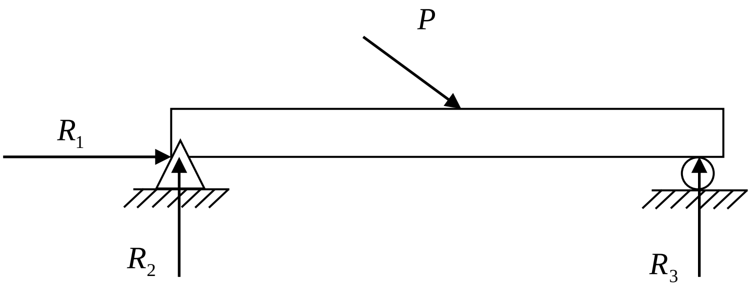
$115\times10^6=\frac{150\times10^3}{A}$

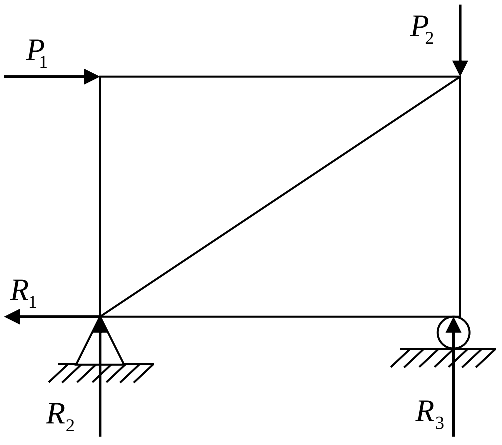
$A=0.0013043\ m^2$

1. $A=t\pi d$
   $0.0013043= t\times\pi\times30\times10^{-3}$
   $t=0.013839\ m=13.839\ mm$
2. $A=t\pi d_2$
   $0.0013043= t\times\pi\times26.353\times10^{-3}$
   $t=0.01575\ m=15.75\ mm$

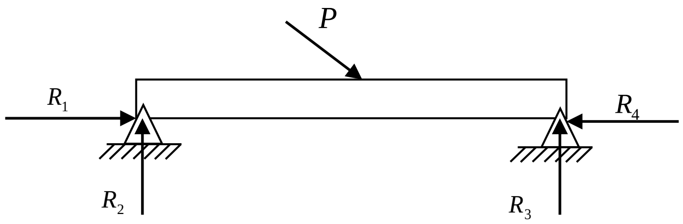
Statically Indeterminate Members:

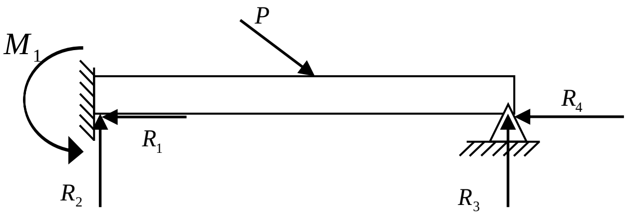
If the values of all the external forces which act on a body can be determined by the equations of static equilibrium alone, then the force system is statically determinate.



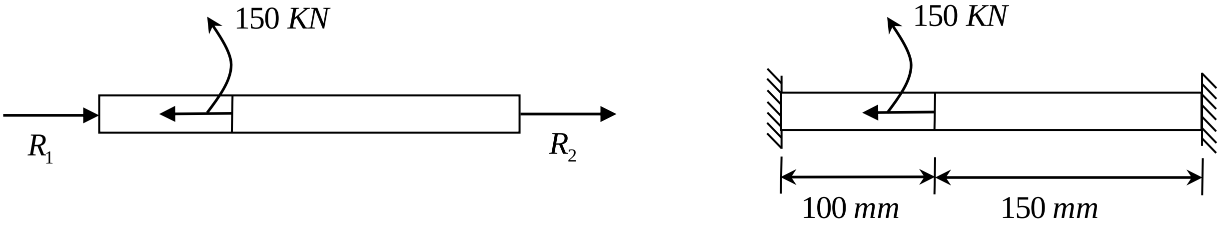


In many cases the forces acting on a body cannot be determined by the equations of static alone because there are more unknown forces than the equations of equilibrium. In such case the force system is said to be statically indeterminate.





Example 11: A square bar ***50 mm*** on a side is held rigidly between the walls and loaded by an axial force of ***150 KN*** as shown. Determine the reactions at the end of the bar and the extension of the right portion. Take ***E=200 GPa***.



$R_1+R_2=150\times10^3$ ………………(1)

$\delta_1=\delta_2$

$$\frac{R_1\times100\times10^{-3}}{50\times10^{-3}\times50\times10^{-3}\times200\times10^{9}}=\frac{R_2\times150\times10^{-3}}{50\times10^{-3}\times50\times10^{-3}\times200\times10^{9}}$$

$0.1R_1=0.15R_2$

$R_1=1.5R_2$ …………………..(2)

From equations (1) and (2)

$1.5R_2+R_2=150\times10^3$

$R_2=60000$ N

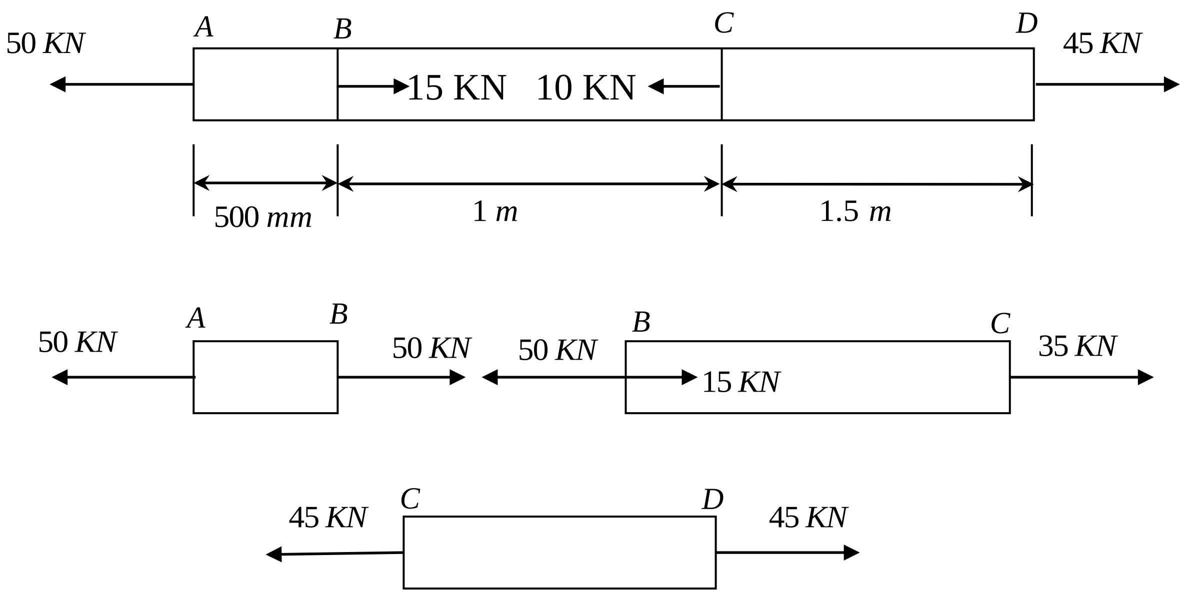
$R_1=90000$ N

$$\delta_2=\frac{R_2\times150\times10^{-3}}{50\times10^{-3}\times50\times10^{-3}\times200\times10^{9}}=\frac{60000\times150\times10^{-3}}{50\times10^{-3}\times50\times10^{-3}\times200\times10^{9}}$$

$\delta_2=0.000018$ m

$\delta_2=0.018$ mm

Example 12: A steel bar of cross section ***500 mm²*** is acted upon by the forces shown. Determine the total elongation of the bar. For steel, ***E=200 GPa***.



❖ ***For portion AB***

$$\delta_1 = \frac{PL}{AE} = \frac{50\times10^{3}\times500\times10^{-3}}{500\times10^{-6}\times200\times10^{9}} = 0.00025\ \text{m} = 0.25\ \text{mm}$$

❖ ***For portion BC***

$$\delta_2 = \frac{PL}{AE} = \frac{35\times10^{3}\times1}{500\times10^{-6}\times200\times10^{9}} = 0.00035\ \text{m} = 0.35\ \text{mm}$$

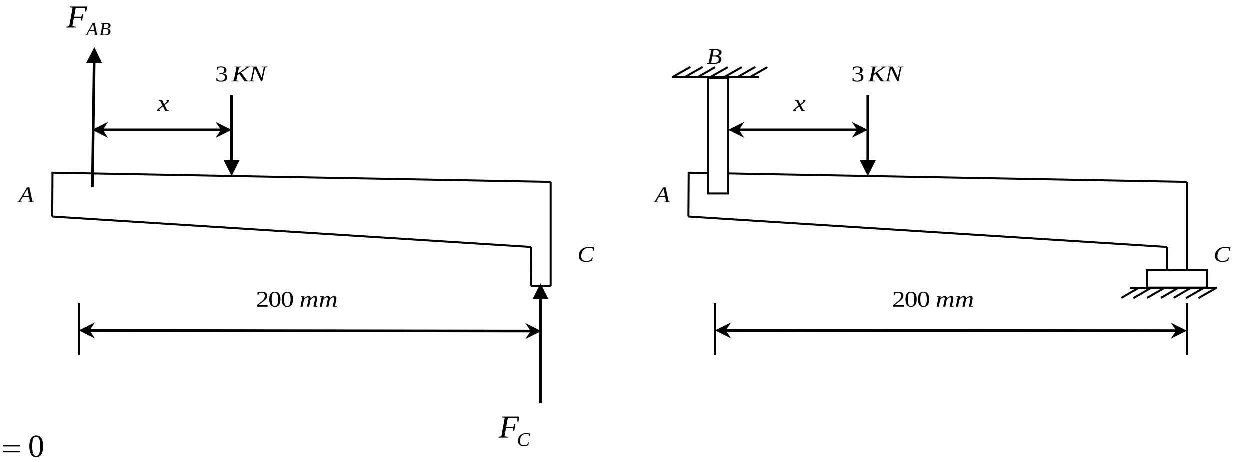
❖ ***For portion CD***

$$\delta_3 = \frac{PL}{AE} = \frac{45\times10^{3}\times1.5}{500\times10^{-6}\times200\times10^{9}} = 0.000675\ \text{m} = 0.675\ \text{mm}$$

$\delta_T = \delta_{1+}\delta_2 + \delta_3$

$\delta_T = 0.25 + 0.35 + 0.675 = 1.275$ mm

Example 13: Member ***AC*** shown is subjected to a vertical force of ***3 KN***. Determine the position ***x*** of this force so that the average compressive stress at ***C*** is equal to the average tensile stress in the tie rod ***AB***. The rod has a cross-sectional area of ***400 mm²*** and the contact area at ***C*** is ***650 mm²***.



$$\sum F_y = 0$$

$F_{AB}+F_C-3000=0$

$F_{AB}+F_C=3000$ ……………………….(1)

$\sigma_{AB}=\sigma_C$

$$\frac{F_{AB}}{A_{AB}} = \frac{F_C}{A_C}$$

$$\frac{F_{AB}}{400\times10^{-6}} = \frac{F_C}{650\times10^{-6}}$$

$F_{AB}=0.6153\ F_C$ …………………….(2)

From equations (1) and (2)

$F_C=1857.24$ N

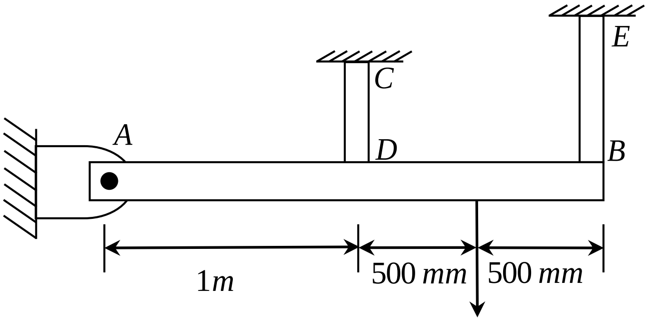
$F_{AB}=1142.759$ N

$$\sum M_A = 0$$

$F_C\times200\times10^{-3}-3000\times x=0$

$x = \dfrac{1857.24\times0.2}{3000}$ =0.123816 m=123.816 mm

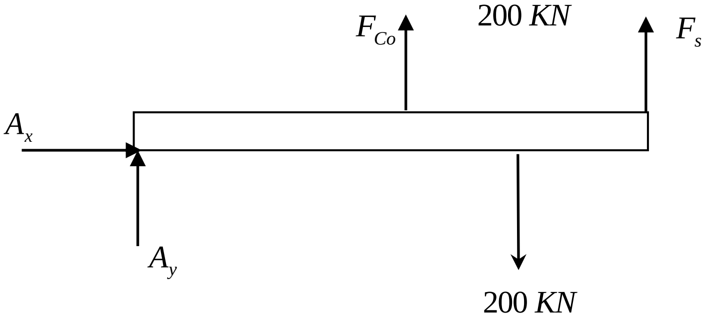
Example 14: The bar ***AB*** is considered to be absolutely rigid and is horizontal before the load of ***200 KN*** is applied. The connection at ***A*** is a pin, and ***AB*** is supported by the steel rod ***EB*** and the copper rod ***CD***. The length of ***CD*** is ***1m***, of ***EB*** is ***2 m***. The cross sectional area of ***CD*** is ***500 mm²***, the area of ***EB*** is ***250 mm²***. Determine the stress in each of the vertical rods and the elongation of the steel rod. Neglect the weight of ***AB***. For copper ***E=120 GPa***, for steel ***E=200 GPa***.



$$\sum M_A = 0$$

$F_{Co}\times1+F_s\times2-200\times10^3\times1.5=0$

$F_{Co}=300\times10^3-2\,F_s$ …………..(1)



$$\frac{\delta_s}{2}=\frac{\delta_{Co}}{1}$$

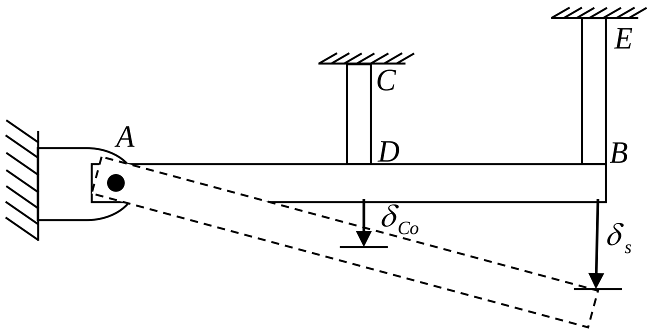
$\delta_s=2\delta_{Co}$

$$\left(\frac{F_s\times L}{A_s E}\right)_s = 2\left(\frac{F_{Co}\times L}{A_{Co} E}\right)_{Co}$$

$$\frac{F_s\times2}{250\times10^{-6}\times200\times10^9}=2\times\frac{F_{Co}\times1}{500\times10^{-6}\times120\times10^9}$$

$F_{Co}=1.2\,F_s$ ……………………..(2)

From equations (1) and (2)



$F_s=93750$ N

$F_{Co}=112500$ N

$$\sigma_s=\frac{F_s}{A_s}=\frac{93750}{250\times10^{-6}}=375000000 \text{ Pa}$$

$\sigma_s=375$ MPa

$$\sigma_{Co}=\frac{F_{Co}}{A_{Co}}=\frac{112500}{500\times10^{-6}}=225000000 \text{ Pa}$$

$\sigma_{Co}=225$ MPa

$$\delta_s=\frac{\sigma_s L}{E}=\frac{375\times10^6\times2}{200\times10^9}=0.00375 \text{ m}=3.75 \text{ mm}$$

Thermal Stresses:

A change in temperature can cause a material to change its dimensions. If the temperature increases, generally a material expands, whereas if the temperature decreases the material will contract.

The deformation of a member having a length L can be calculated using the formula:

$$\delta_T=\alpha\times\Delta T\times L$$

$$\delta_T=\frac{FL}{AE}=\alpha\times\Delta T\times L$$

$$\sigma_T=E\times\alpha\times\Delta T$$

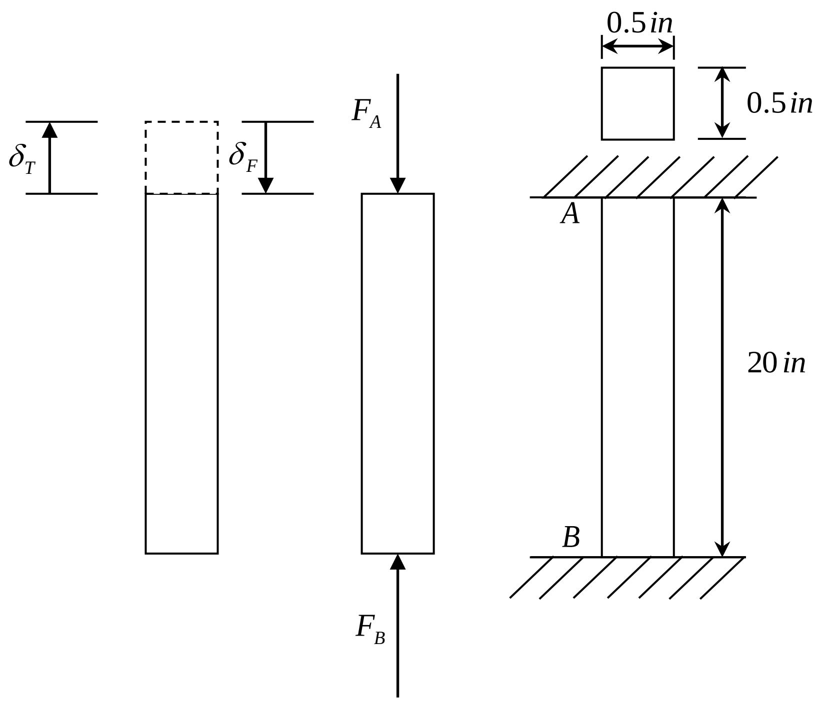
α: Linear coefficient of thermal expansion. The units measure strain per degree of temperature. They are (1/ºF) in the foot-pound-second system and (1/ºC) or (1/ºK) in SI system.

ΔT: Change in temperature of the member.

L: The original length of the member.

δ_T: The change in length of the member.

Example 15: The A-36 steel bar shown is constrained to just fit between two fixed supports when $\boldsymbol{T_1=60^o\ F}$. If the temperature is raised to $\boldsymbol{T_2=120^o\ F}$ determine the average normal thermal stress developed in the bar. For steel $\boldsymbol{\alpha=6.6\times10^{-6}\ 1/^oF}$, $\boldsymbol{E=29\times10^3\ Ksi}$.



$$\sum F_y=0$$

$F_A-F_B=F$

$\delta_T-\delta_F=0$

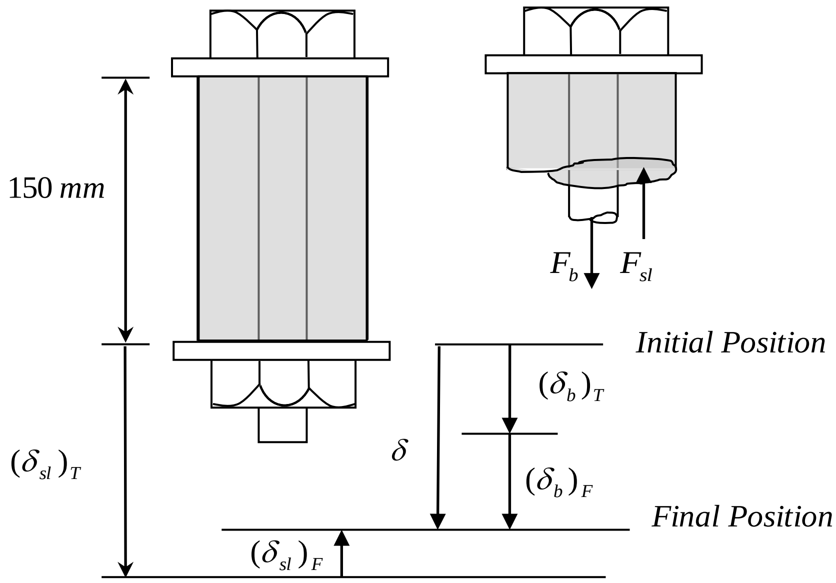
$\delta_T=\alpha\times\Delta T\times L$

$\sigma_T=E\times\alpha\times\Delta T$

$=29\times10^3\times6.6\times10^{-6}\times(120-60)$

$=11.484$ Ksi

Example 16: A 2014-T6 aluminum tube having a cross sectional area of ***600 mm²*** is used as a sleeve for an A-36 steel bolt having a cross sectional area of ***400 mm²***. When the temperature is ***T_1=15° C***, the nut hold the assembly in a snug position such that the axial force in the bolt is negligible. If the temperature increases ***T_2=80° C***, determine the average normal stress in the bolt and sleeve. For aluminum ***α=23×10⁻⁶ 1/°C***, ***E=73.1 GPa,*** for steel ***α=12×10⁻⁶ 1/°C***, ***E=200 GPa.***



$$\sum F_y = 0$$

$F_{sl}-F_b=0$

$F_{sl}=F_b=F$

$$\delta = (\delta_b)_T + (\delta_b)_F = (\delta_{sl})_T - (\delta_{sl})_F$$

$$\left[\alpha\times\Delta T\times L+\frac{FL}{AE}\right]_b=\left[\alpha\times\Delta T\times L-\frac{FL}{AE}\right]_{sl}$$

$$12\times10^{-6}\times0.15\times(80-15)+\frac{F\times0.15}{400\times10^{-6}\times200\times10^{9}} = 23\times10^{-6}\times0.15\times(80-15)-\frac{F\times0.15}{600\times10^{-6}\times73.1\times10^{9}}$$

$0.0052949\times10^{-6}F=0.00010725$

$F=20255$ N

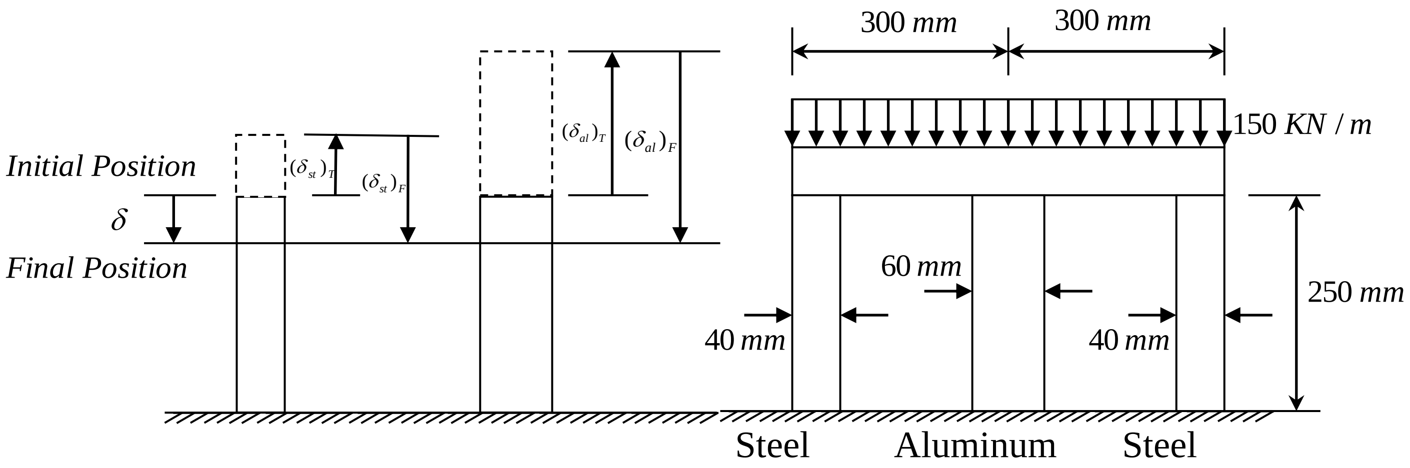
$$\sigma_b=\frac{F}{A_b}=\frac{20255}{400\times10^{-6}}$$

$\sigma_b=50.637655$ MPa

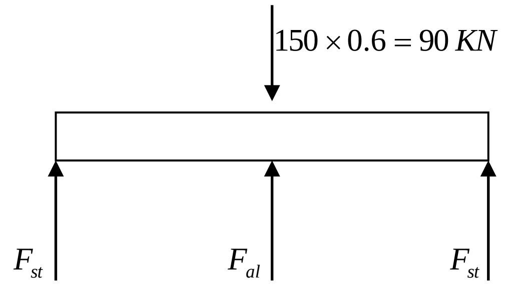
$$\sigma_{sl}=\frac{F}{A_{sl}}=\frac{20255}{600\times10^{-6}}$$

$\sigma_{sl}=33.758436$ MPa

Example 17: The rigid bar shown is fixed to the top of the three posts made of A-36 steel and 2014-T6 aluminum. The posts each have a length of ***250 mm*** when no load is applied to the bar, and the temperature is $\boldsymbol{T_1=20^{\circ}C}$. Determine the force supported by each posts if the bar is subjected to a uniform distributed load of ***150 KN/m*** and the temperature is raised to $\boldsymbol{T_2=80^{\circ}C}$. For steel $\boldsymbol{\alpha=12\times10^{-6}\ 1/^{\circ}C}$, ***E=200 GPa*** , for aluminum $\boldsymbol{\alpha=23\times10^{-6}\ 1/^{\circ}C}$, ***E=73.1 GPa***.



$$\sum F_y = 0$$



$2F_{st}+F_{al}=90000$ …………….(1)

$\delta=(\delta_{st})_T-(\delta_{st})_F=(\delta_{al})_T-(\delta_{al})_F$

$$\left[\alpha\times\Delta T\times L-\frac{F_{st}L}{AE}\right]_{st}=\left[\alpha\times\Delta T\times L-\frac{F_{al}L}{AE}\right]_{al}$$

$$12\times10^{-6}\times0.25\times(80-20)-\frac{F_{st}\times0.25}{\frac{\pi}{4}(40\times10^{-3})^2\times200\times10^9}=23\times10^{-6}\times0.25\times(80-20)-\frac{F_{al}\times0.25}{\frac{\pi}{4}(60\times10^{-3})^2\times73.1\times10^9}$$

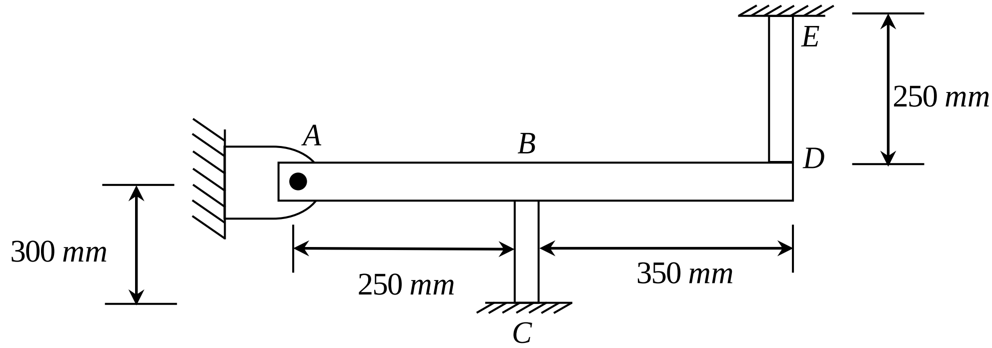
$1.20956\times10^{-9}F_{al}-0.994718\times10^{-9}F_{st}=0.000165$ ………………..(2)

From equations (1) and (2)

$F_{st}=-16444.7$ N

$F_{al}=122888.8$ N

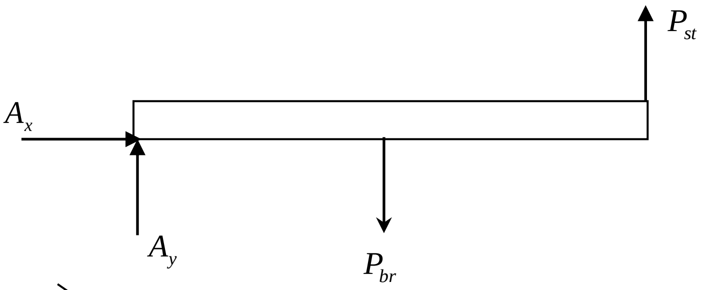
Example 18: The rigid bar ***AD*** is pinned at ***A*** and attached to the bars ***BC*** and ***ED*** as shown. The entire system is initially stress-free and the weights of all bars are negligible. The temperature of bar ***BC*** is lowered ***25ºK*** and that of the bar ***ED*** is raised ***25ºK***. Neglecting any possibility of lateral buckling, find the normal stresses in bars ***BC*** and ***ED***. For ***BC***, which is brass, assume ***E=90 GPa***, $\boldsymbol{\alpha=20\times10^{-6}\ 1/^{o}K}$ and for ***ED***, which is steel, take $\boldsymbol{\alpha=12\times10^{-6}\ 1/^{o}K}$, ***E=200 GPa***. The cross-sectional area of ***BC*** is ***500 mm²***, of ***ED*** is ***250 mm²***.



$$\sum M_A = 0$$

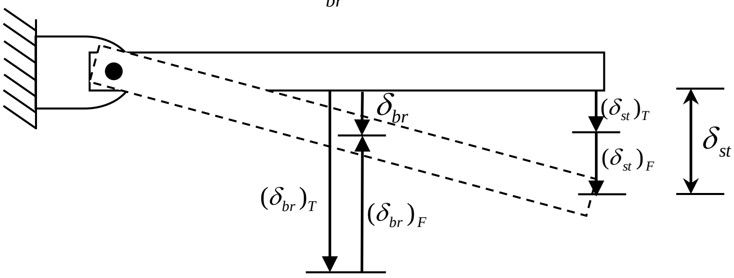
$P_{st}\times600\times10^{-3}-P_{br}\times250\times10^{-3}=0$

$P_{st}=0.41666\ P_{br}$ ………..(1)



$$\frac{\delta_{br}}{250}=\frac{\delta_{st}}{600}$$

$$\frac{\alpha\times L\times\Delta T-\dfrac{P_{br}\times L}{A_{br}E_{br}}}{250}=\frac{\alpha\times L\times\Delta T+\dfrac{P_{st}\times L}{A_{st}E_{st}}}{600}$$



$$\frac{20\times10^{-6}\times300\times10^{-3}\times25-\dfrac{P_{br}\times300\times10^{-3}}{500\times10^{-6}\times90\times10^{9}}}{250}=\frac{12\times10^{-6}\times250\times10^{-3}\times25+\dfrac{P_{st}\times250\times10^{-3}}{250\times10^{-6}\times200\times10^{9}}}{600}$$

$8.333\times10^{-12}\ P_{st}+26.666\times10^{-12}\ P_{br}=475\times10^{-9}$ …………..(2)

From equations (1) and (2)

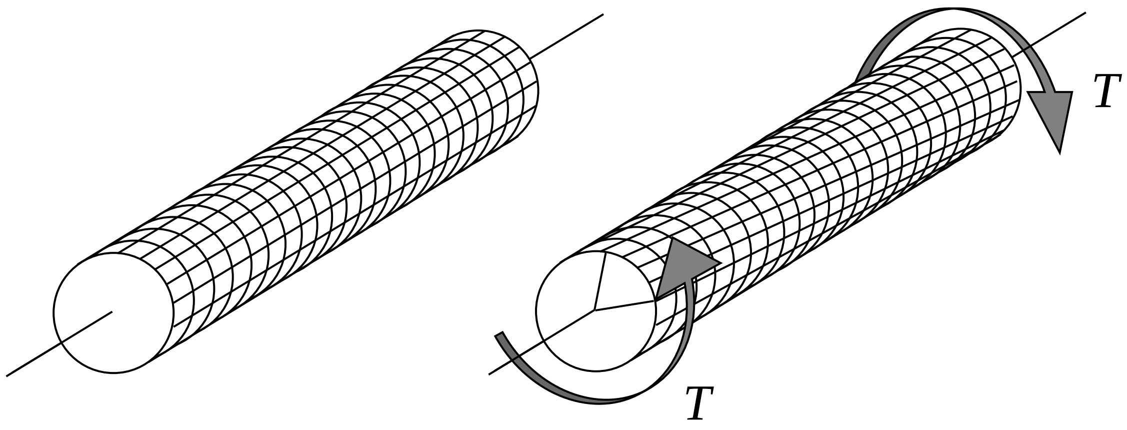
$P_{br}=15760.5$ N , $P_{st}=6566.77$ N

$$\sigma_{br}=\frac{15760.5}{500\times10^{-6}}=31.521MPa$$

$$P_{st}=\frac{6566.77}{250\times10^{-6}}=26.267MPa$$

Torsion:

Torque is a moment that tends to twist a member about its longitudinal axis. When the torque is applied, the circles and longitudinal grid lines originally marked on the shaft tend to distort into the pattern shown below.

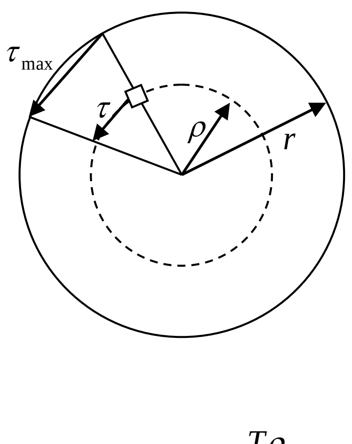


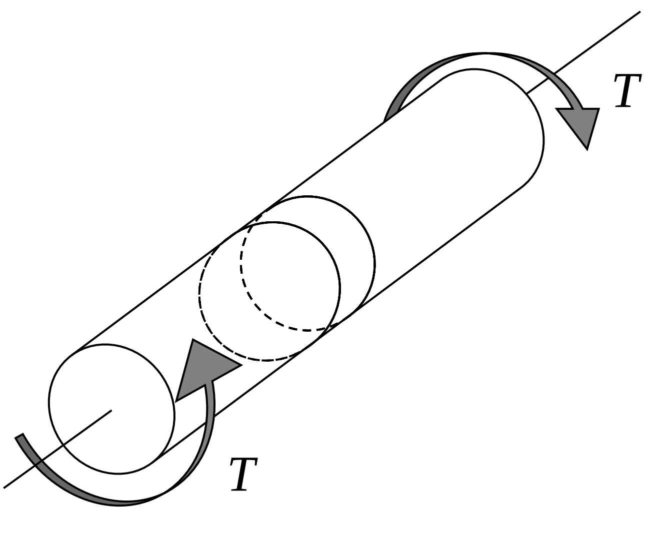
Before deformation After deformation

Twisting causes the circles to remain circles and each longitudinal grid line deforms into a helix that intersects the circles at equal angles. Also, the cross sections at the ends of the shaft remain flat that is, they do not warp or bulge in or out and radial lines on these ends remain straight during the deformation.

The Torsion Formula:

Consider a uniform circular shaft is subjected to a torque it can be shown that every section of the shaft is subjected to a state of pure shear.





$$\tau = \frac{T\rho}{J}$$

τ: The torsional shearing stress.
T: The resultant internal torque acting on the cross section.
ρ: The distance from the centre (radial position).
J: The polar moment of inertia of the cross sectional area.

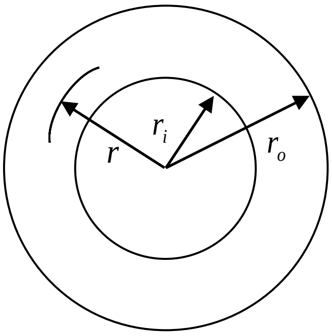
$$\tau_{\max} = \frac{Tr}{J}$$

τ_{max}: The maximum shear stress in the shaft, which occurs at the outer surface.
r: The outer radius of the shaft.

$$J = \frac{\pi}{2} r^4$$

$$J = \frac{\pi}{32} D^4$$

for a hollow shaft



$$\tau = \frac{Tr}{J}$$

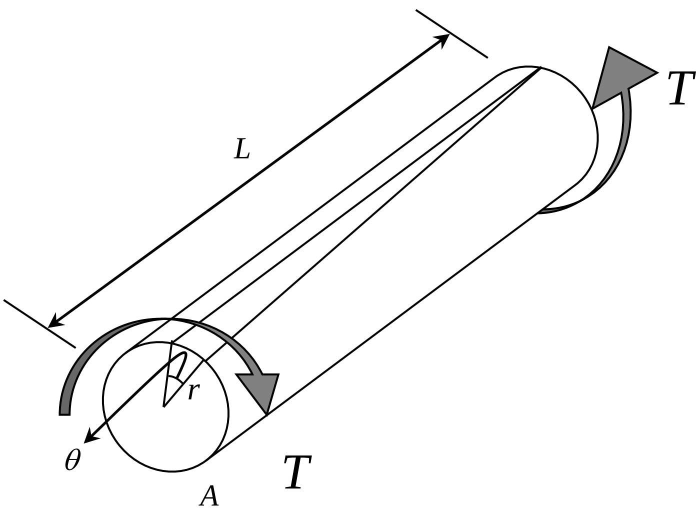
$$\tau_{\max} = \frac{Tr_o}{J}$$

$$J = \frac{\pi}{2}(r_o^4 - r_i^4) = \frac{\pi}{32}(D_o^4 - D_i^4)$$

Angle of Twist (θ):

If a shaft of length L is subjected to a constant twisting moment along its length, then the angle of twist θ through which one end of the shaft will twist relative to the other is:

$$\theta = \frac{TL}{GJ}$$



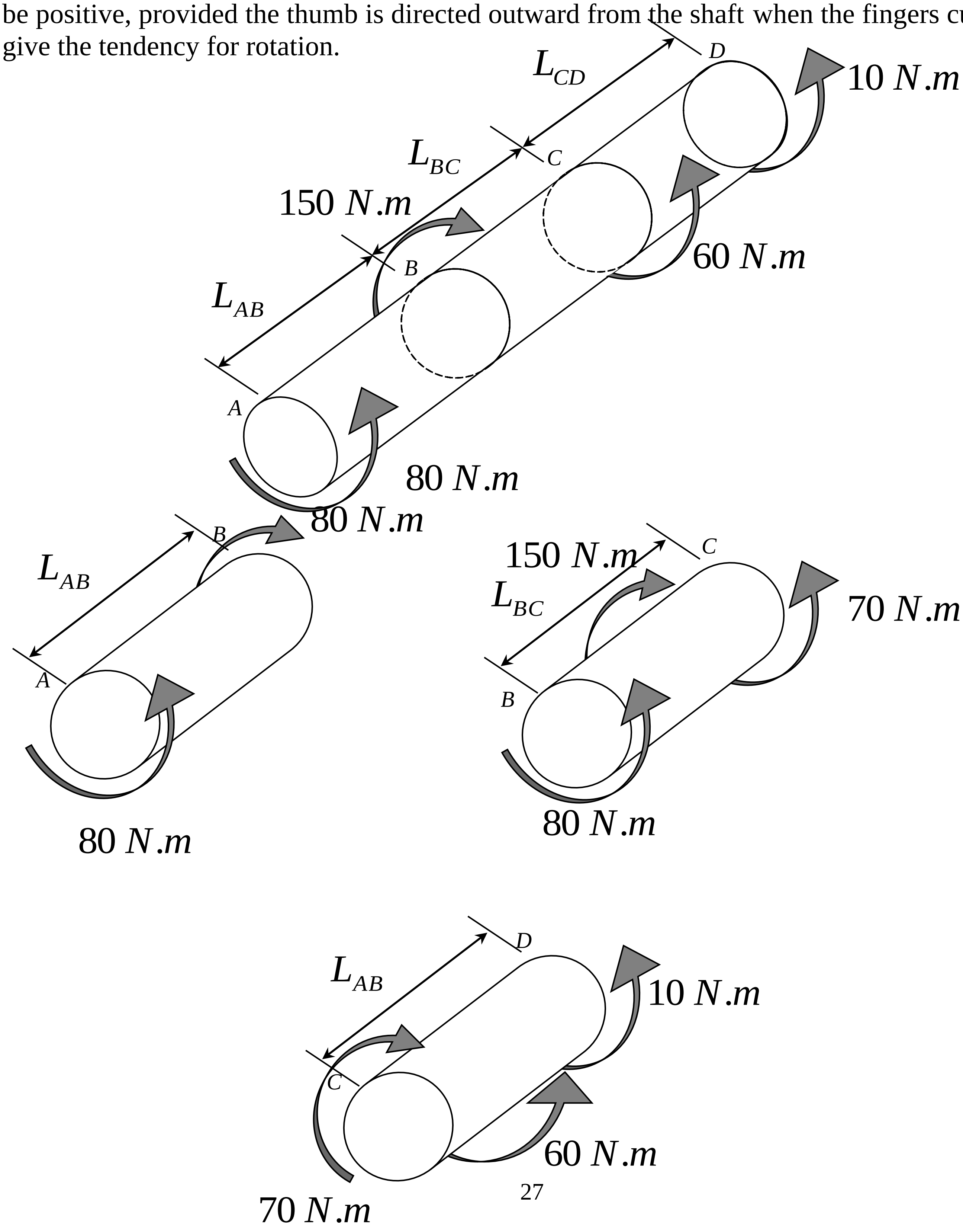
G: The shear modulus of elasticity or modulus of rigidity.

θ: Angle of twist, measured in rad

If the shaft is subjected to several different torques or the cross sectional area or shear modulus changes from one region to the next. The angle of twist of one end of the shaft with respect to the other is then found from:

$$\theta = \sum \frac{TL}{GJ}$$

In order to apply the above equation, we must develop a sign convention for the internal torque and the angle of twist of one end of the shaft with respect to the other end. To do this, we will use the right hand rule, whereby both the torque and angle of twist will be positive, provided the thumb is directed outward from the shaft when the fingers curl to give the tendency for rotation.



$$\theta_{A/D} = \frac{80 \times L_{AB}}{GJ} - \frac{70 \times L_{BC}}{GJ} - \frac{10 \times L_{CD}}{GJ}$$

Power Transmission (P):

Shaft and tubes having circular cross sections are often used to transmit power developed by a machine.

$$P = T \times \frac{d\theta}{dt} \quad , \ \frac{d\theta}{dt} = \omega$$

ω: The shaft's angular velocity (rad/s).

$$P = T \times \omega$$

In SI units power is expressed in (watts) when torque is measured in (N.m) and ω in (rad/s).

1 W=1 N.m/s

In the foot-pound-second or FPS system the units of power are (ft.lb/s); however horsepower (hp) is often used in engineering practice where:

1 hp=550 ft.lb/s

For machinery the frequency of a shaft's rotation f is often reported. This is a measured of the number of revolutions or cycles the shaft per second and is expressed in hertz (1 Hz=1 cycle/s), 1 cycle=2π rad, then ω=2πf

P=2πfT

Example 19: If a twisting moment of ***1 KN.m*** is impressed upon a ***50 mm*** diameter shaft, what is the maximum shearing stress developed? Also what is the angle of twist in a ***1 m*** length of the shaft? The material is steel, for which ***G=85 GPa***.

$$\tau_{\max} = \frac{Tr}{J}$$

$$J = \frac{\pi}{32} D^4 = \frac{\pi}{32} (50 \times 10^{-3})^4 = 0.6135 \times 10^{-6} \ \text{m}^4$$

$$\tau_{\max} = \frac{1 \times 10^3 \times 25 \times 10^{-3}}{0.6135 \times 10^{-6}}$$

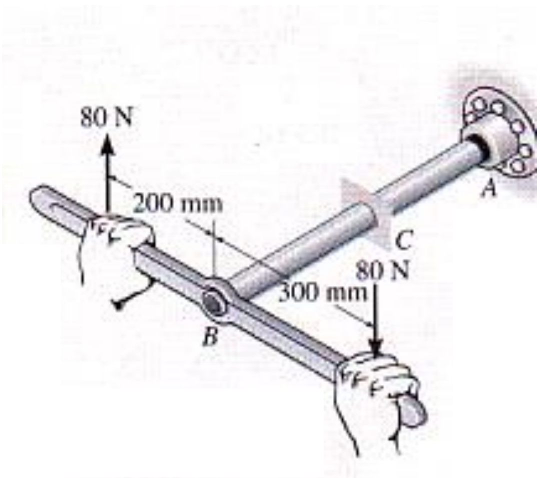
τ_{max}=40.74979 MPa

$$\theta = \frac{TL}{GJ}$$

$$\theta = \frac{1\times 10^{3}\times 1}{85\times 10^{9}\times 0.6135\times 10^{-6}}$$

θ=0.01917 rad.

Example 20: The pipe shown has an inner diameter of ***80 mm*** and an outer diameter of ***100 mm***. If its end is tightened against the support at ***A*** using a torque wrench at ***B***, determine the shear stress developed in the material at the inner and outer walls along the central portion of the pipe when the ***80 N*** forces are applied to the wrench.



$T=80\times 200\times 10^{-3}+80\times 300\times 10^{-3}$

T=40 N.m

$$\tau = \frac{Tr}{J}$$

$$J = \frac{\pi}{32}(D_o^4 - D_i^4)$$

$$J = \frac{\pi}{32}\left[(100\times 10^{-3})^4 - (80\times 10^{-3})^4\right]$$

$J=5.7962\times 10^{-6}$ m^4

- Inner walls r_i=40 mm

$$\tau = \frac{Tr_i}{J} = \frac{40\times 40\times 10^{-3}}{5.7962\times 10^{-6}}$$

τ=0.276042 MPa

- Outer walls r_o=50 mm

$$\tau = \frac{Tr_o}{J} = \frac{40\times 50\times 10^{-3}}{5.7962\times 10^{-6}}$$

τ=0.345053 MPa

Example 21: The gear motor can developed ***0.1 hp*** when it turns at ***80 rev/min***. If the allowable shear stress for the shaft is $\boldsymbol{\tau_{allow}=4\ ksi}$, determine the smallest diameter of the shaft that can be used.

$$\tau_{allow} = \frac{Tr}{J}$$

$$J = \frac{\pi}{2} r^4$$

$$\tau_{allow} = \frac{Tr}{\frac{\pi}{2} r^4}$$

$$\tau_{allow} = \frac{2T}{\pi r^3}$$

$$r = \sqrt[3]{\frac{2T}{\pi \tau_{allow}}}$$

P=T.ω $\Longrightarrow$ $T = \dfrac{P}{\omega}$

P=0.1×550=55 lb/s

ω=80×2π/60=8.377 rad/s

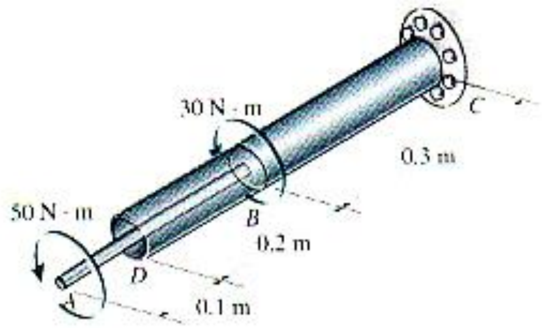
$T = \dfrac{55}{8.377}$ =6.5655 lb.ft

T=6.5655×12=78.786 lb.in

$r = \sqrt[3]{\dfrac{2 \times 78.786}{\pi \times 4 \times 10^3}}$ =0.2323 in

d=0.4646 in

Example 22: The assembly consists of a solid ***15 mm*** diameter rod connected to the inside of a tube using a rigid disk at ***B***. Determine the absolute maximum shear stress in the rod and in the tube. The tube has an outer diameter of ***30 mm*** and a wall thickness of ***3 mm***.



- The rod

  T=50 N.m

  r=7.5×10^{-3} m

$$J=\frac{\pi}{2}r^4$$

$$J=\frac{\pi}{2}(75\times10^{-3})^4$$

J=4.97009×10^{-9} m^4

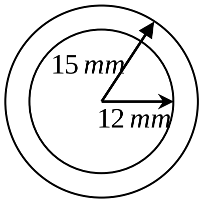
$$\tau_r=\frac{Tr}{J}$$

$$\tau_r=\frac{50\times7.5\times10^{-3}}{4.97009\times10^{-9}}\text{=75.4512 MPa}$$

- The tube

  T=80 N.m

  r=15×10^{-3} m

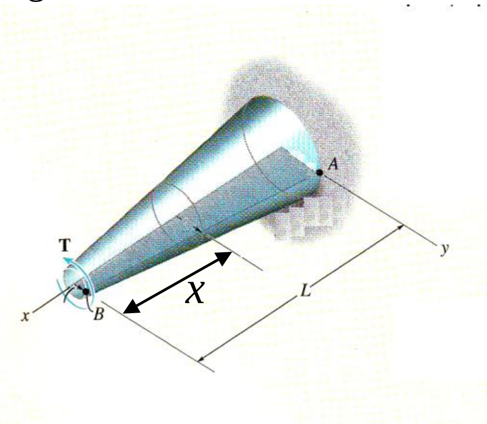


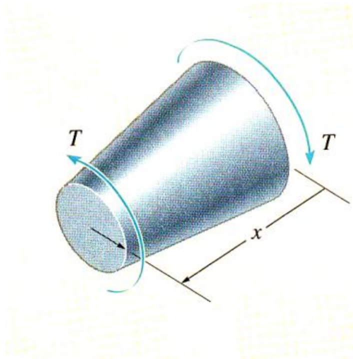
$$J=\frac{\pi}{2}(r_o^4-r_i^4)$$

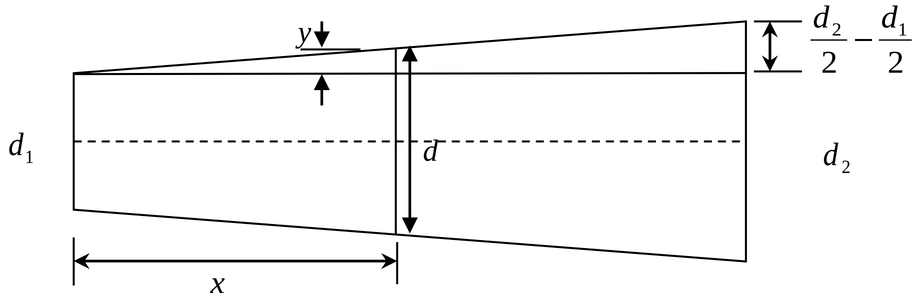
$$J=\frac{\pi}{2}\left[(15\times10^{-3})^4-(12\times10^{-3})^4\right]\text{=46.9495×10}^{-9}\text{ m}^4$$

$$\tau_t=\frac{Tr_o}{J}=\frac{80\times15\times10^{-3}}{46.9495\times10^{-9}}\text{=25.559 MPa}$$

Example 23: The tapered shaft shown below is made of a material having a shear modulus G. Determine the angle of twist of its end B when subjected to the torque.







$$\frac{y}{x} = \frac{d_2 - d_1}{2L} \Rightarrow y = \frac{d_2 - d_1}{2}\frac{x}{L}$$

d=d$_1$+2y=d$_1$+$(d_2 - d_1)\frac{x}{L}$

$$J(x) = \frac{\pi}{32}d^4 = \frac{\pi}{32}[d_1 + (d_2 - d_1)\frac{x}{L}]^4$$

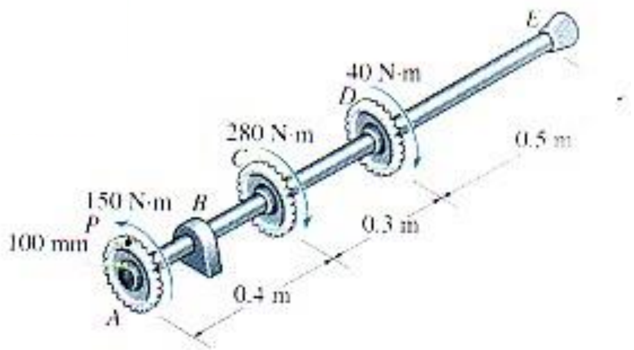
$$\varphi = \int_0^L \frac{T\,dx}{G\,J(x)} = \int_0^L \frac{T\,dx}{G\,\frac{\pi}{32}[d_1 + (d_2 - d_1)\frac{x}{L}]^4} = \frac{32T}{\pi G}\int_0^L \frac{dx}{[d_1 + (d_2 - d_1)\frac{x}{L}]^4}$$

$$\varphi = \frac{32TL}{3\pi G(d_2 - d_1)}\left[\frac{1}{[d_1 + (d_2 - d_1)\frac{x}{L}]^3}\right]_0^L = \frac{32TL}{3\pi G(d_2 - d_1)}\left(\frac{1}{d_2^3} - \frac{1}{d_1^3}\right)$$

$$\varphi = \frac{32TL}{3\pi G(d_2 - d_1)}\left(\frac{d_1^3 - d_2^3}{d_2^3 . d_1^3}\right)$$

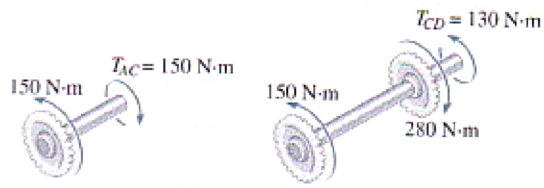
$$\varphi = \frac{32TL}{3\pi G}\left(\frac{d_1^2 + d_1 d_2 + d_2^2}{d_2^3 . d_1^3}\right)$$

Example 24: The gears attached to the fixed end steel shaft are subjected to the torques shown. If the shear modulus of elasticity is ***G=80 GPa*** and the shaft has a diameter of ***14 mm***, determine the displacement of the tooth ***P*** on gear ***A***.



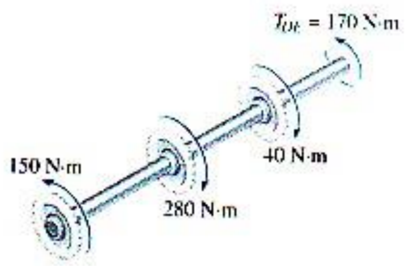
- Segment AC

$150-T_{AC}=0$
$T_{AC}=150$ N.m



- Segment CD

$150-280+T_{CD}=0$
$T_{CD}=130$ N.m



- Segment DE

$-130-40+T_{DE}=0$
$T_{DE}=170$ N.m

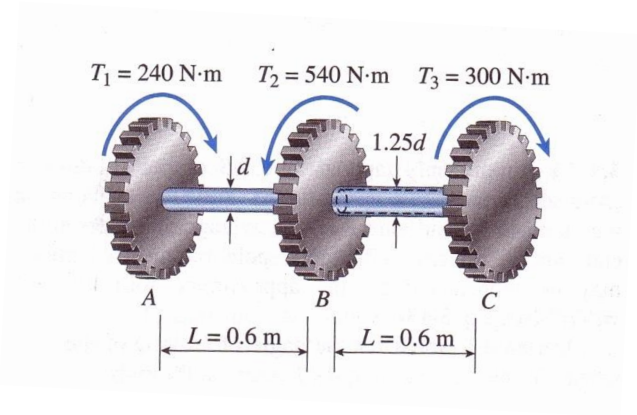
$$\theta=\sum\frac{TL}{GJ}$$

$$\theta=\frac{1}{GJ}\sum TL$$

$$=\frac{1}{80\times10^{9}\times\frac{\pi}{2}(7\times10^{-3})^{4}}\left[150\times0.4-130\times0.3-170\times0.5\right]$$

θ=-0.21211 rad

Example 25: A steel shaft ***ABC*** connecting three gears consists of a solid bar of diameter ***d*** between gears ***A*** and ***B*** and a hollow bar of outside diameter ***1.25d*** and inside diameter ***d*** between gears ***B*** and ***C***. Both bars have length ***0.6 m***. The gears transmit torques ***T_1=240 N.m***, ***T_2=540 N.m***, and ***T_3=300 N.m*** acting in the directions shown in the figure. The shear modulus of elasticity for the shaft is ***80 GPa***. ***a)*** what is the minimum permissible diameter ***d*** if the allowable shear stress in the shaft is ***80 MPa***?. ***b)*** what is the minimum permissible diameter ***d*** if the angle of twist between any two gears is limited to ***4°***?.



T_{AB}=240 N.m
T_{BC}=300 N.m

a)

1. ***For solid bar AB***

J=$\frac{\pi}{32}d^4$

$$\tau = \frac{T_{AB}d/2}{J} = \frac{T_{AB}d/2}{\frac{\pi}{32}d^4}$$

$$80\times10^6 = \frac{240\times d/2}{\frac{\pi}{32}d^4}$$

$$d^3 = \frac{16\times240}{\pi\times80\times10^6}$$

d=0.02481 m
d=24.81 mm

2. ***For hollow bar BC***

J=$\frac{\pi}{32}[d_o^4 - d_i^4] = \frac{\pi}{32}[(1.25d)^4 - d^4]$

$$\tau = \frac{T_{BC}\times1.25\times d/2}{J} = \frac{T_{BC}\times1.25\times d/2}{\frac{\pi}{32}[(1.25d)^4 - d^4]}$$

$$80\times10^6 = \frac{300\times1.25\times d/2}{\frac{\pi}{32}[(1.25d)^4 - d^4]} \Longrightarrow d^3 = \frac{16\times300\times1.25}{\pi\times80\times10^6\times1.4414}$$

d=0.0255 m
d=25.5 mm **Answer**

b)

1. ***For solid bar AB***

$$\theta = \frac{T_{AB}L}{GJ}$$

$$4\times\frac{\pi}{180}=\frac{240\times0.6}{80\times10^{9}\times\frac{\pi}{32}d^{4}}$$

d=0.02263 m
d=22.63 mm

2. For hollow bar BC

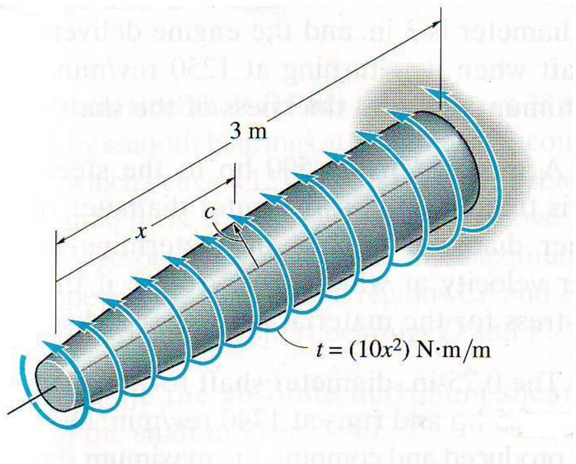
$$\theta=\frac{T_{BC}L}{GJ}$$

$$4\times\frac{\pi}{180}=\frac{300\times0.6}{80\times10^{9}\times\frac{\pi}{32}[(1.25d)^{4}-d^{4}]}$$

d=0.02184 m
d=21.84 mm

d=0.02263 m
d=22.63 mm ***Answer***

Example 26:_ The shaft is subjected to a distributed torque along its length of ***t=10x²* *N.m/m***, where ***x*** is in meters. If the maximum stress in the shaft is t_o remain constant at ***80 MPa***, determine the required variation of the radius ***r*** of the shaft for $0\le x\le 3$ ***m***



$$T=\int t dx$$

$$=\int 10x^{2}dx=\frac{10}{3}x^{3}$$

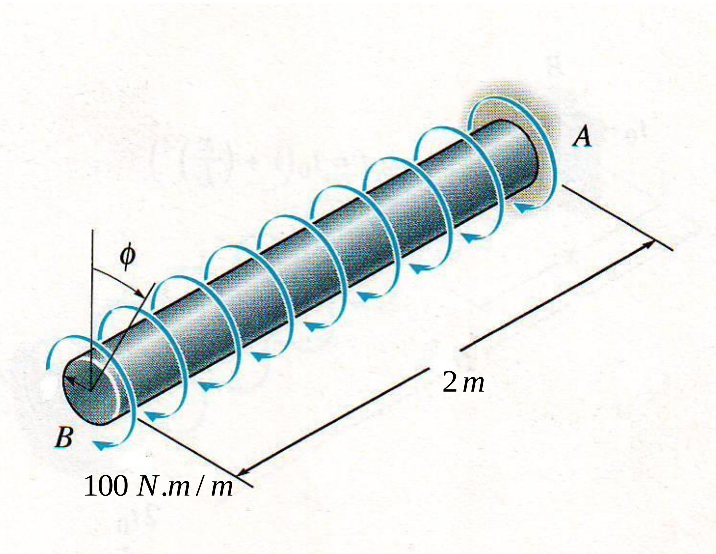
$$\tau_{\max}=\frac{Tr}{J}$$

$$80\times10^{6}=\frac{\frac{10}{3}x^{3}r}{\frac{\pi}{2}r^{4}}\quad\Rightarrow\quad 80\times10^{6}=\frac{20x^{3}}{3\pi r^{3}}$$

$$r = \sqrt[3]{\frac{20x^3}{3\pi \times 80 \times 10^6}} = 0.002982x \text{ m}$$

r=2.982 x mm

Example 26: The shaft has a radius ***50 mm*** and is subjected to a torque per unit length of ***100 N.m*** which is distributed uniformly over the shafts entire length ***2 m***. If it is fixed at its far end ***A***, determine the angle of twist of end ***B***. The shear modulus is ***73.1 GPa***.



T(x)=100x

$$\theta = \int_0^2 \frac{T(x)dx}{GJ} = \int_0^2 \frac{100xdx}{73.1\times 10^9 \times \frac{\pi}{2}(50\times 10^{-3})^4}$$

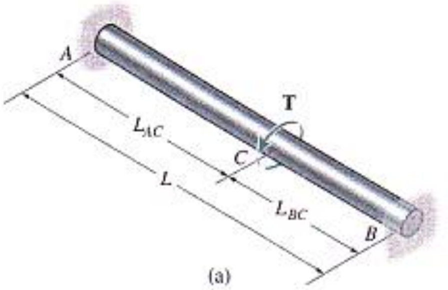
$$= \frac{200}{73.1\times 10^9 \times \pi(50\times 10^{-3})^4}\left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{400}{73.1\times 10^9 \times \pi(50\times 10^{-3})^4}$$

$$\theta = 2.786\times 10^{-4} \text{ rad}$$

$=0.01596^{o}$

Statically Indeterminate:



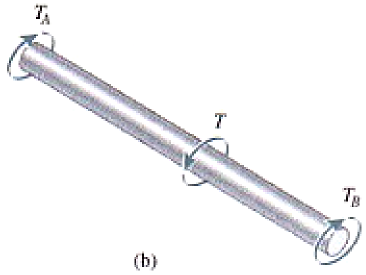
(a)

$T_A - T + T_B = 0$

$T_A + T_B = T$

$\theta_{A/B} = 0$

$$\frac{T_A L_{AC}}{GJ} - \frac{T_B L_{BC}}{GJ} = 0$$

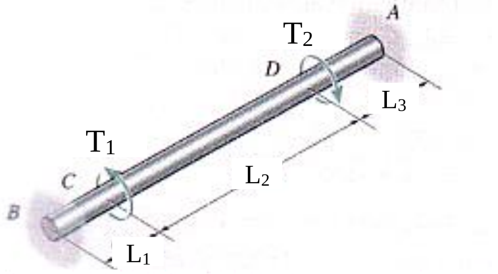


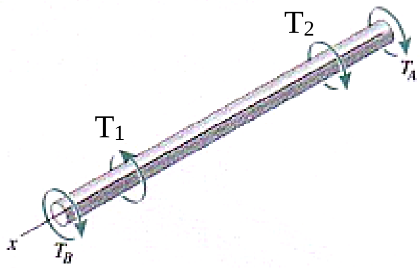
(b)

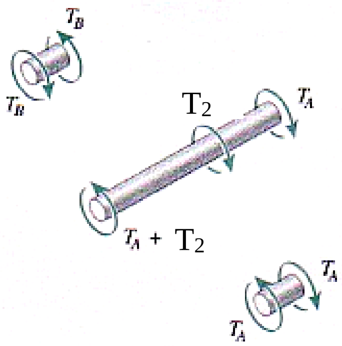
If $T_1 > T_2$

$-T_A - T_2 + T_1 - T_B = 0$

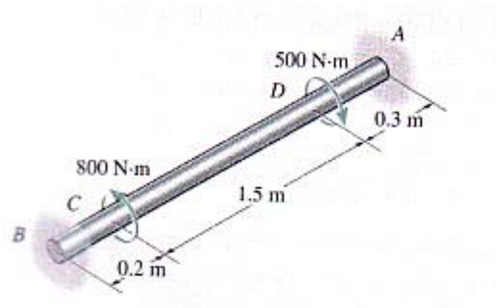
$$\frac{-T_B L_1}{GJ} + \frac{(T_A + T_2) L_2}{GJ} + \frac{T_A L_3}{GJ} = 0$$

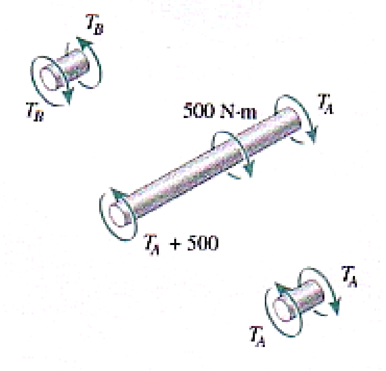


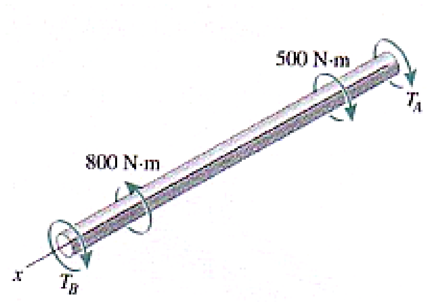




Example 27: The solid steel shaft shown has a diameter of ***20 mm***. If it is subjected to the two torques, determine the reactions at the fixed supports ***A*** and ***B***.







$-T_B+800-500-T_A=0$

$T_B+T_A=300$ ……………………………………....(1)

$$\frac{-T_B \times 0.2}{GJ}+\frac{(T_A+500)\times 1.5}{GJ}+\frac{T_A \times 0.3}{GJ}=0$$

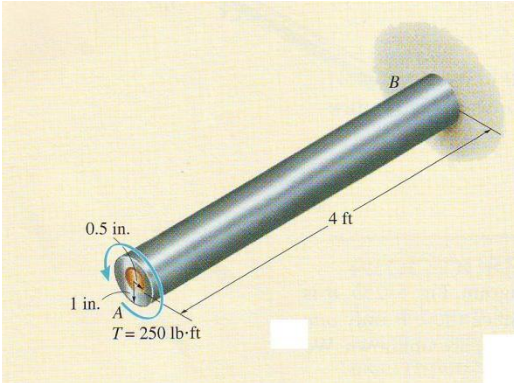
$-0.2T_B+1.5T_A+750+0.3T_A=0$

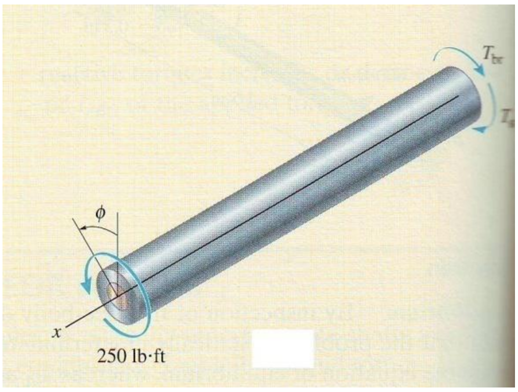
$1.8T_A-0.2T_B=-750$ …………………………...…(2)

$T_A=-345$ N.m

$T_B=645$ N.m

Example 28: The shaft shown below is made from steel tube, which is bonded to a brass core. If a torque of T=250 lb.ft is applied at its end, plot the shear stress distribution along a radial line of its cross sectional area. Take $G_{st}11.4\times10^3$ ksi, $G_{br}=5.2\times10^3$ ksi.





$T_{st}+T_{br}=250\times12=3000$ lb.in…………(1)

$\theta=\theta_{st}=\theta_{br}$

$$\frac{T_{st}L}{11.4\times10^6\times\frac{\pi}{2}[(1)^4-(0.5)^4]}=\frac{T_{br}L}{5.2\times10^6\times\frac{\pi}{2}[(1)^4-(0.5)^4]}$$

$T_{st}=32.88T_{br}$………………………(2)

From (1) and (2)

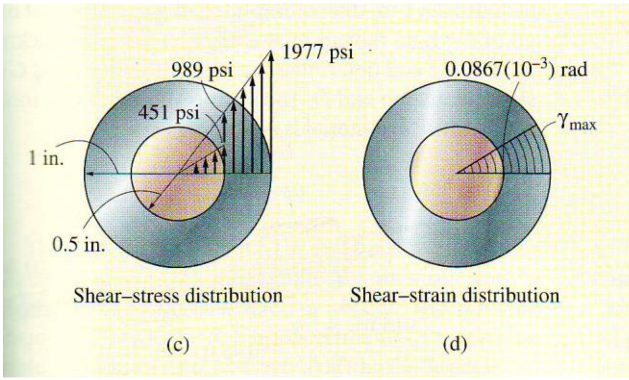
$T_{st}=2911$ lb.in=242.6 lb.ft

$T_{br}=88.5$ lb.in=7.38 lb.ft

$$(\tau_{br})_{\max}=\frac{88.5\times0.5}{\frac{\pi}{2}(0.5)^4}=451\quad psi$$

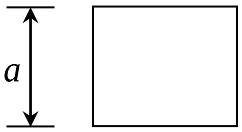
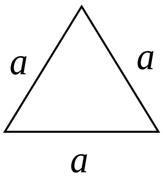
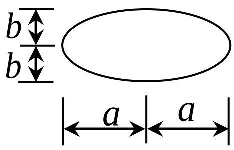
$$(\tau_{st})_{\max}=\frac{2911\times1}{\frac{\pi}{2}[(1)^4-(0.5)^4]}=1977\quad psi$$

$$(\tau_{st})_{\min}=\frac{2911\times0.5}{\frac{\pi}{2}[(1)^4-(0.5)^4]}=988\quad psi$$

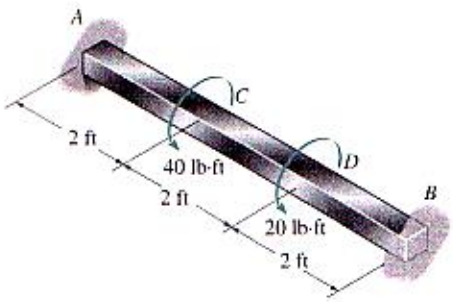


$$\gamma=\frac{\tau}{G}=\frac{451}{5.2\times10^6}=\frac{988}{11.4\times10^6}=0.0867\times10^{-3}\quad rad$$

<u>Torsion of Solid Noncircular Shafts:</u>

Shape of cross section	τ_{max}	θ
 Square	$\dfrac{4.81T}{a^3}$	$\dfrac{7.1TL}{a^4G}$
 Equilateral triangle	$\dfrac{20T}{a^3}$	$\dfrac{46TL}{a^4G}$
 Ellipse	$\dfrac{2T}{\pi ab^2}$	$\dfrac{(a^2+b^2)TL}{\pi a^3b^3G}$

Example 28: The 2014-T6 aluminum strut is fixed between the two walls at ***A*** and ***B***. If it has a ***2 in*** by ***2 in*** square cross section and it is subjected to the torsional loading shown, determine the reactions at the fixed supports. Also what is the angle of twist at ***C***. Take ***G=3.9×103 ksi***.



$T_A - 40 - 20 + T_B = 0$

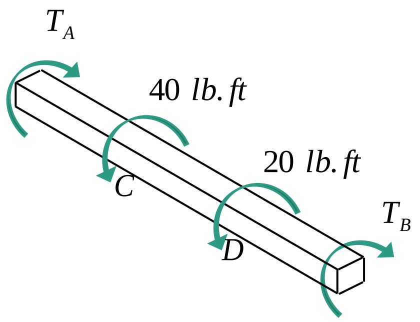
$T_A + T_B = 60$ ……….(1)

$\theta_{A/B} = 0$

$$\theta_{A/B} = \sum \frac{7.1TL}{a^4G}$$

$$\frac{7.1T_A\times12\times2\times12}{2^4\times3.9\times10^6} - \frac{7.1(T_B-20)\times12\times2\times12}{2^4\times3.9\times10^6} - \frac{7.1T_B\times12\times2\times12}{2^4\times3.9\times10^6} = 0$$

$T_A - 2T_B = -20$ ………………(2)



From equations (1) and (2)

$T_B = 26.666$ lb.ft
$T_A = 33.333$ lb.ft

$$\theta_C = \frac{7.1 T_A L}{a^4 G}$$

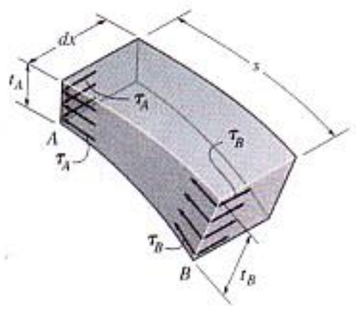
$$\theta_C = \frac{7.1 \times 33.333 \times 12 \times 2 \times 12}{2^4 \times 3.9 \times 10^6}$$

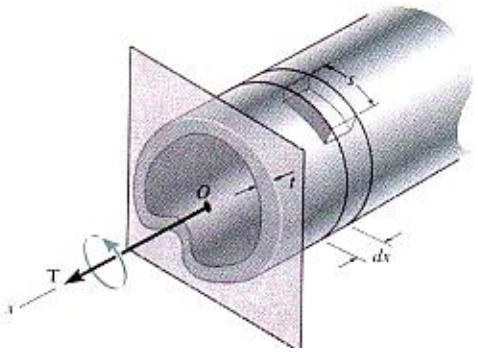
$\theta_C = 0.001092$ rad

$\theta_C = 0.06258^\circ$

Thin walled tubes having closed cross sections:

Shear flow(q): is the product of the tube's thickness and the average shear stress. This value is constant at all points along the tube's cross section. As a result, the largest average shear stress on the cross section occurs where the tube's thickness is smallest.



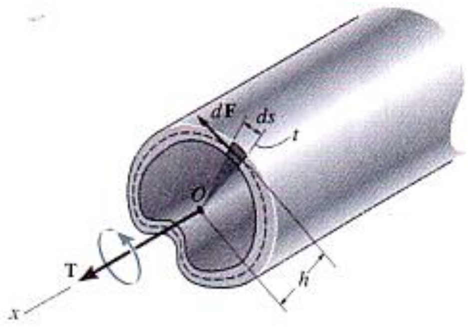


The forces acting on the two faces are $dF_A = \tau_A (t_A dx)$, $dF_B = \tau_B (t_B dx)$, these forces are equal for equilibrium, so that:

$\tau_A t_A = \tau_B t_B$
$q = \tau_{avg} t$

Average shear stress(τ_{avg}):

The average shear stress acting on the shaded Area dA=tds



(e)

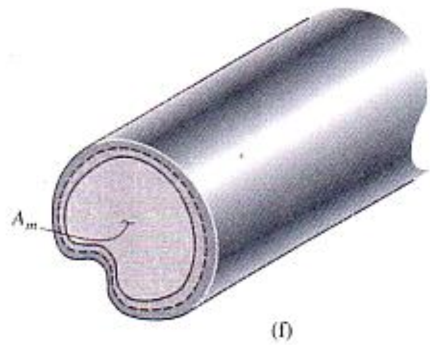
$$dF=\tau_{avg}dA=\tau_{avg}tds$$

$$dT=dF\times h=\tau_{avg}tds\times h$$

$$T=\tau_{avg}t\int hds$$

Area of triangle $dA_m=\frac{1}{2}hds$

$$hds=2dA_m$$



(f)

$$T=2\tau_{avg}t\int dA_m=2\tau_{avg}tA_m$$

$$\tau_{avg}=\frac{T}{2tA_m}$$

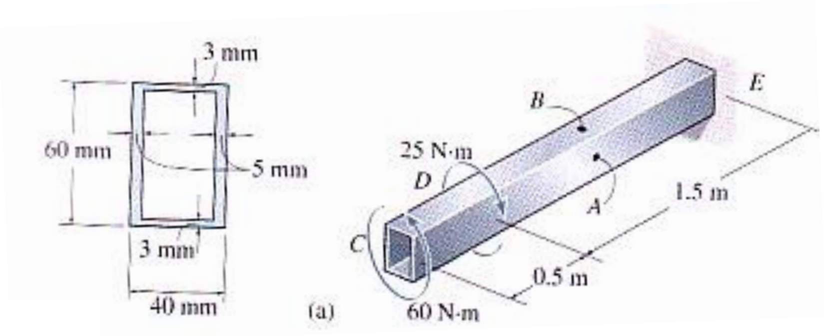
τ_{avg}: The average shear stress acting over the thickness of the tube.
T: The resultant internal torque at the cross section.
t: The thickness of the tube where τ_{avg} is to be determined.
A_m: The mean area enclosed within the boundary of the center line of the tube thickness.

$$q=\tau_{avg}t=\frac{T}{2A_m}$$

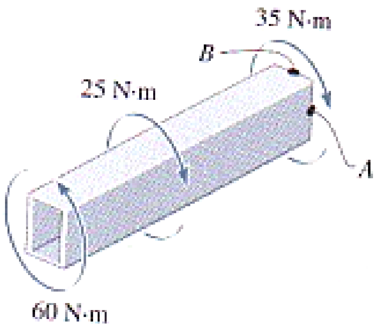
Angle of Twist(θ):

$$\theta=\frac{TL}{4A_m^2G}\oint\frac{ds}{t}$$

Example 29: The tube is made of ***C86100*** bronze and has a rectangular cross section as shown below. If its subjected to the two torques, determine the average shear stress in the tube at points ***A*** and ***B***. Also, what is the angle of twist of end ***C***? The tube is fixed at ***E***. Take ***G=38 GPa***.



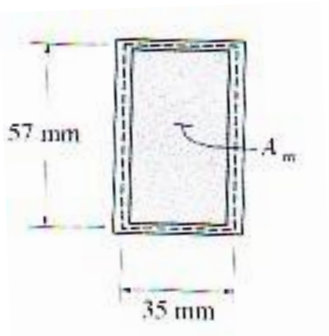
60-25-T=0
T=35 N.m



$A_m=(40\times10^{-3}-5\times10^{-3})(60\times10^{-3}-3\times10^{-3})$
$=0.001995\ m^2$

$$\tau_A=\frac{T}{2tA_m}=\frac{35}{2\times5\times10^{-3}\times0.001995}$$

$\tau_A=1.7543859$ MPa



$$\tau_B=\frac{T}{2tA_m}=\frac{35}{2\times3\times10^{-3}\times0.001995}$$

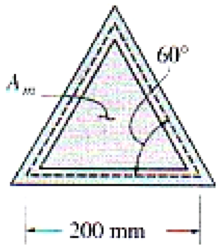
$\tau_B=2.9239766$ MPa

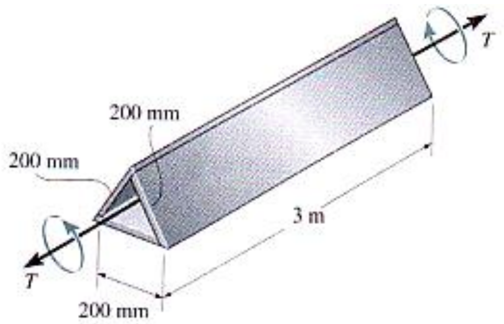
$$\theta=\sum\frac{TL}{4A_m^2G}\oint\frac{ds}{t}$$

$$\theta=\frac{60\times0.5}{4\times(0.001995)^2\times3.8\times10^9}[2\times\frac{35\times10^{-3}}{3\times10^{-3}}+2\times\frac{57\times10^{-3}}{5\times10^{-3}}]$$
$$+\frac{35\times1.5}{4\times(0.001995)^2\times3.8\times10^9}[2\times\frac{35\times10^{-3}}{3\times10^{-3}}+2\times\frac{57\times10^{-3}}{5\times10^{-3}}]$$

$\theta=0.0062912$ rad

Example 30: A thin tube is made from three ***5 mm*** thick ***A-36*** steel plates such that it has a cross section that is triangular as shown below. Determine the maximum torque ***T*** to which it can be subjected, if the allowable shear stress is τ_{allow}***=90 MPa*** and the tube is restricted to twist no more than $\theta=2\times10^{-3}$ ***rad***. Take ***G=75 GPa***.





$$A=\frac{1}{2}(200\times10^{-3})\times(200\times10^{-3}\sin60)=0.01732\ m^2$$

$$t=0.005\ m$$

$$\tau_{allow}=\frac{T}{2tA_m}=\frac{T}{2\times5\times10^{-3}\times0.01732}=90\times10^6$$

$$T=15.588\ KN.m$$

$$\theta=\frac{TL}{4A_m^2G}\oint\frac{ds}{t}$$

$$2\times10^{-3}=\frac{T\times3}{4\times(0.01732)^2\times75\times10^9}[3\times\frac{200\times10^{-3}}{5\times10^{-3}}]$$

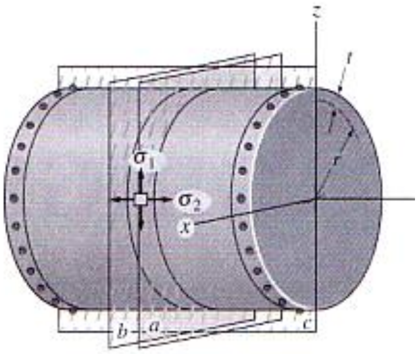
$$T=500\ N.m$$

Thin Walled Cylinder, Thin Walled Pressure Vessels:

Cylindrical or spherical vessels are commonly used in industry to serve as boilers or tanks. When under pressure, the material of which they are made is subjected to a loading from all directions. In general "thin wall" refers to a vessel having an inner radius to wall thickness ratio of 10 or more ($r/t \geq 10$)

1. Cylindrical Vessels:

Consider the cylindrical vessel having a wall thickness t and inner radius r as shown below. A pressure p is developed within the vessel by a containing gas or fluid, which is assumed to have negligible weight.

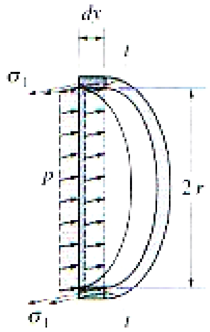


The stresses set up in the walls are:

a. Circumferential or hoop stress

$2[\sigma_1(tdy)]-p(2rdy)=0$

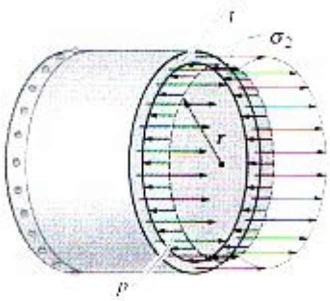
$$\sigma_1 = \frac{pr}{t}$$



b. Longitudinal or axial stress

$\sigma_2(2\pi rt)-p(\pi r^2)=0$

$$\sigma_2 = \frac{pr}{2t}$$



c. Circumferential or hoop strain

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2)$$

d. Longitudinal strain

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1)$$

e. Change in length

The change in length of the cylinder may be determined from the longitudinal strain.

Change in length=longitudinal strain×original length

$\delta L = \varepsilon_2 L = \frac{1}{E}(\sigma_2 - \nu\sigma_1)\,L$

$\delta L = \frac{pr}{2tE}(1 - 2\nu)\,L$

f. Change in diameter

The change in diameter may be found from the circumferential change.

Change in diameter=diametral strain×original diameter

Diametral strain=circumferential strain

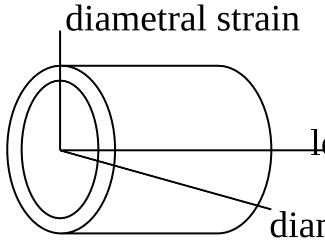
$\delta d = \varepsilon_1 d = \frac{1}{E}(\sigma_1 - \nu\sigma_2)\,d$

$\delta d = \frac{pr}{2tE}(2 - \nu)\,d$

g. Change in internal volume

Volumetric strain=longitudinal strain+2diametral strain

$\varepsilon_v = \varepsilon_2 + 2\varepsilon_1 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) + 2\frac{1}{E}(\sigma_1 - \nu\sigma_2)$



$\varepsilon_v = \frac{1}{E}(\sigma_2 - \nu\sigma_1 + 2\sigma_1 - 2\nu\sigma_2)$

$= \frac{1}{E}\left(\frac{pr}{2t} - \nu\frac{pr}{t} + 2\frac{pr}{t} - \nu\frac{pr}{t}\right)$

$\varepsilon_v = \frac{pr}{2tE}(5 - 4\nu)$

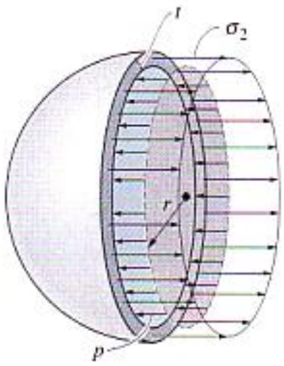
change in internal volume=volumetric strain×original volume

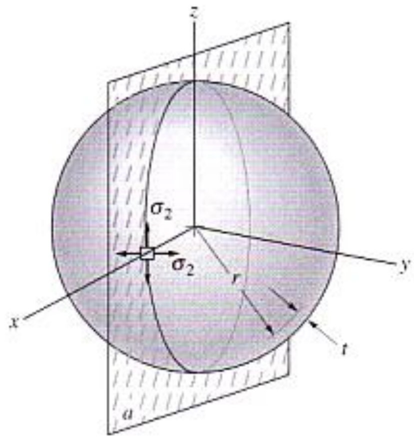
$\delta v = \varepsilon_v v$

$\delta v = \frac{pr}{2tE}(5 - 4\nu)\,v$

2. Spherical Vessels:

Because of the symmetry of the sphere the stresses set up owing to internal pressure will be two mutually perpendicular hoop or circumferential stress of equal value and a radial stress.





$$\sigma_1(2\pi rt)-p(\pi r^2)=0$$

$$\sigma_1 = \frac{pr}{2t}$$

$$\sigma_2 = \sigma_1 = \frac{pr}{2t}$$

Change in internal volume

change in internal volume=volumetric strain×original volume
volumetric strain=3hoop strain

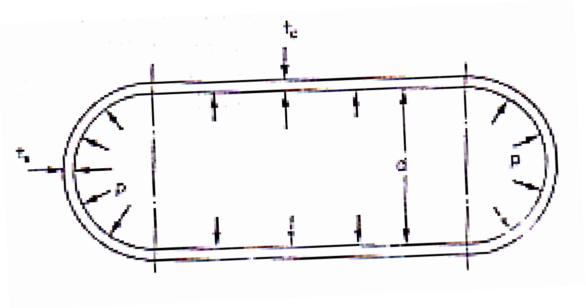
$$\varepsilon_v = \varepsilon_1 = 3\frac{1}{E}(\sigma_1 - \nu\sigma_2) = \frac{3\sigma_1}{E}(1-\nu) = \frac{3pr}{2tE}(1-\nu)$$

$$\delta v = \varepsilon_v v$$

$$\delta v = \frac{3pr}{2tE}(1-\nu)\,v$$

Cylindrical Vessels with Hemispherical Ends:

r=d/2



a) For the cylindrical portion

$$\sigma_1 = \frac{\Pr}{t_c} \quad \text{hoop stress}$$

$$\sigma_2 = \frac{\Pr}{2t_c} \quad \text{longitudinal stress}$$

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) = \frac{1}{E}(\frac{\Pr}{t_c} - \nu\frac{\Pr}{2t_c})$$

$$\varepsilon_1 = \frac{pr}{2t_c E}(2-\nu) \qquad \text{hoop strain}$$

b) For the spherical ends

$$\sigma_1 = \frac{\Pr}{2t_s} \qquad \text{hoop stress}$$

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) = \frac{\sigma_1}{E}(1-\nu)$$

$$\varepsilon_1 = \frac{pr}{2t_s E}(1-\nu) \qquad \text{hoop strain}$$

Thus equating the two strains in order that there shall be no distortion of the junction.

$$\frac{pr}{2t_s E}(1-\nu) = \frac{pr}{2t_c E}(2-\nu)$$

$$\frac{t_s}{t_c} = \frac{1-\upsilon}{2-\upsilon}$$

Example 31: A thin cylinder ***75 mm*** internal diameter, ***250 mm*** long with walls ***2.5 mm*** thick is subjected to an internal pressure of ***7 MN/m²***. Determine the change in internal diameter and the change in length. If in addition to the internal pressure, the cylinder is subjected to a torque of ***200 N.m*** find the magnitude and nature of the stresses set up in the cylinder.***E=200 GN/m², υ=0.3***.

$$\delta d = \frac{pr}{2tE}(2-\nu)\,d$$

$$\delta d = \frac{7\times10^{6}\times\frac{75}{2}\times10^{-3}}{2\times2.5\times10^{-3}\times200\times10^{9}}[2-0.3]\times75\times10^{-3}$$

$\delta d = 33.468\times10^{-6}$ m $= 33.468$ μm

$$\delta L = \frac{pr}{2tE}(1-2\nu)\,L$$

$$\delta L = \frac{7\times10^{6}\times\frac{75}{2}\times10^{-3}}{2\times2.5\times10^{-3}\times200\times10^{9}}[1-2\times0.3]\times250\times10^{-3}$$

$\delta L = 26.25\times10^{-6}$ m $= 26.25$ μm

$$\sigma_1 = \frac{pr}{t} = \frac{7\times10^{6}\times\frac{75}{2}\times10^{-3}}{2.5\times10^{-3}}$$

$\sigma_1 = 105\times10^{6}$ N/m² $= 105$ MN/m²

$$\sigma_2 = \frac{pr}{2t} = \frac{7\times10^{6}\times\frac{75}{2}\times10^{-3}}{2\times2.5\times10^{-3}}$$

$\sigma_2 = 52.5\times10^{6}$ N/m² $= 52.5$ MN/m²

$$\tau = \frac{Tr}{J} = \frac{Tr}{\frac{\pi}{2}[r_o^4 - r_i^4]} = \frac{200\times40\times10^{-3}}{\frac{\pi}{2}[(40\times10^{-3})^4 - (37.5\times10^{-3})^4]}$$

$\tau = 8.743862$ MN/m²

Example 32: A cylinder has an internal diameter of ***230 mm***, has walls ***5 mm*** thick and is ***1 m*** long. It is found to change in internal volume by ***12×10⁻⁶ m³*** when filled with a liquid at a pressure ***p***. If ***E=200 GN/m²*** and ***υ=0.25***, and assuming rigid end plates, determine ***a)*** the values of hoop and longitudinal stresses ***b)*** the necessary change in pressure ***p*** to produce a further increase in internal volume of ***15%***.

a) $\delta v = \frac{pr}{2tE}(5-4\nu)\,v$

$$12\times10^{-6} = \frac{p\times\frac{230}{2}\times10^{-3}}{2\times5\times10^{-3}\times200\times10^{9}}[5-4\times0.25]\times\pi\times(\frac{230}{2}\times10^{-3})^{2}\times1$$

$p=1.255763$ MN/m^2

$$\sigma_1 = \frac{pr}{t} = \frac{1.255763\times10^{6}\times\frac{230}{2}\times10^{-3}}{5\times10^{-3}}$$

$\sigma_1=28.882549$ MN/m^2

$$\sigma_2 = \frac{pr}{2t} = \frac{1.255763\times10^{6}\times\frac{230}{2}\times10^{-3}}{2\times5\times10^{-3}}$$

$\sigma_2=14.4412745$ MN/m^2

b) $\delta v=1.15\times12\times10^{-6}=13.8\times10^{-6}$ m^3

$\delta v = \frac{pr}{2tE}(5-4\nu)\,v$

$$13.8\times10^{-6} = \frac{p\times\frac{230}{2}\times10^{-3}}{2\times5\times10^{-3}\times200\times10^{9}}[5-4\times0.25]\times\pi\times(\frac{230}{2}\times10^{-3})^{2}\times1$$

$p=1.444128$ MN/m^2

Necessary increase=1.444128-1.255763=0.188365 MN/m^2

Vessels Subjected to Fluid Pressure:

If a fluid is used as the pressurization medium the fluid itself will change in volume as pressure is increased and this must be taken into account when calculating the amount of fluid which must be pumped into the cylinder in order to raise the pressure by a specific amount.

The bulk modulus of a fluid is defined as:

bulk modulus k= $\dfrac{Volumetric stress}{Volumetric strain}$

volumetric stress=pressure p

volumetric strain= $\dfrac{change in volume}{original volume} = \dfrac{\delta v}{v}$

k= $\dfrac{p}{\frac{\delta v}{v}} = \dfrac{pv}{\delta v}$

change in volume of fluid under pressure= $\dfrac{pv}{k}$

extra fluid required to raise cylinder pressure by p

$$= \frac{pr}{2tE}(5-4\nu)\,\mathrm{v} + \frac{pv}{k}$$

extra fluid required to raise sphere pressure by p

$$= \frac{3pr}{2tE}(1-\nu)\,\mathrm{v} + \frac{pv}{k}$$

Example 33: a) A sphere 1m internal diameter and ***6 mm*** wall thickness is to be pressure tested for safety purposes with water as the pressure medium. Assuming that the sphere is initially filled with water at atmospheric pressure, what extra volume of water is required to be pumped in to produce a pressure of ***3 MN/m²*** gauge? For water ***k=2.1 GN/m²*** ***b)*** The sphere is now placed in service and filled with gas until there is a volume change of ***72×10⁻⁶ m³***. Determine the pressure exerted by the gas on the walls of the sphere. ***c)*** To what value can the gas pressure be increased before failure occurs according to the maximum principal stress theory of elastic failure? ***E=200 GPa***, ***υ=0.3*** and the yield stress is simple tension***=280 MPa***.

a) extra volume of water $=\frac{3pr}{2tE}(1-\nu)\,\mathrm{v}+\frac{pv}{k}$

$$=\frac{3\times3\times10^{6}\times0.5}{2\times6\times10^{-3}\times200\times10^{9}}(1-0.3)\times\frac{4}{3}\pi(0.5)^{3}+\frac{3\times10^{6}\times\frac{4}{3}\pi(0.5)^{3}}{2.1\times10^{9}}$$

$=0.001435221\ \mathrm{m}^3$

b) $\delta\mathrm{v}=\frac{3pr}{2tE}(1-\nu)\,\mathrm{v}$

$$72\times10^{-6}=\frac{3p\times0.5}{2\times6\times10^{-3}\times200\times10^{9}}(1-0.3)\times\frac{4}{3}\pi(0.5)^{3}$$

$p=0.31430827\ \mathrm{MN/m^2}$

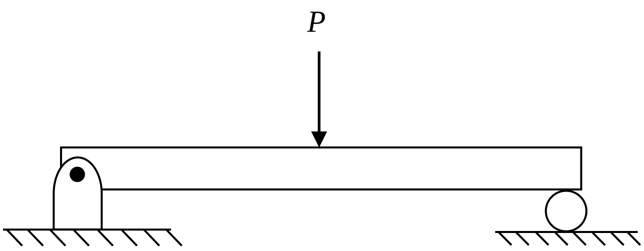
$\sigma_1=\frac{pr}{2t}$ σ_1=yield stress for maximum principal stress theory

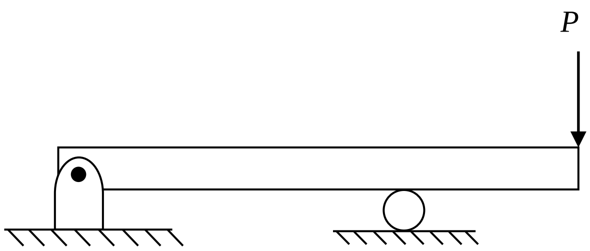
$$280\times10^{6}=\frac{p\times0.5}{2\times6\times10^{-3}}$$

$p=6.72\ \mathrm{MN/m^2}$

Shear and Moment Diagram:

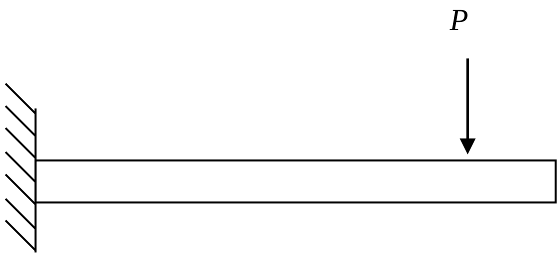
Beams are long straight members that carry loads perpendicular to their longitudinal axis. They are classified according to the way they are supported, e.g. simply supported, cantilevered, or overhanging.





Simply supported beam

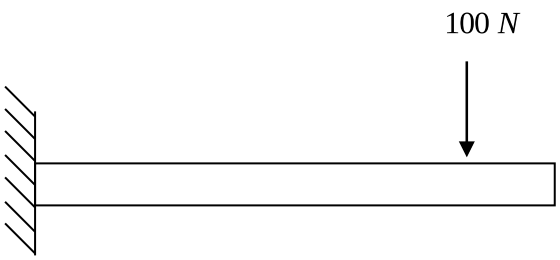
overhanging beam



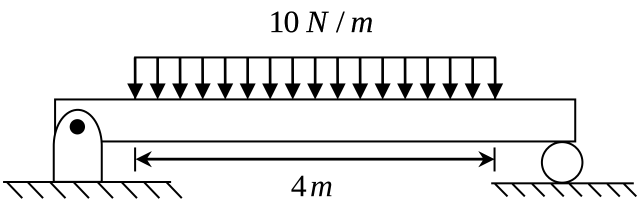
Cantilevered beam

Types of Loading:

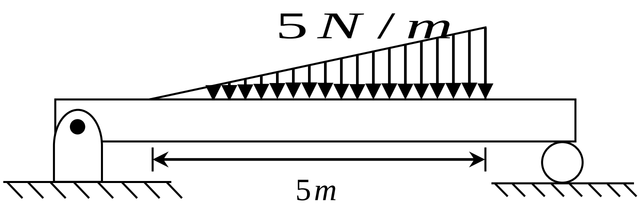
Loads commonly applied to a beam may consist of concentrated forces(applied at a point), uniformly distributed loads, in which case the magnitude is expressed as a certain number of newtons per meter of length of the beam, or uniformly varying loads. A beam may also be loaded by an applied couple.



(point load)
(concentrated force)



Uniformly distributed load



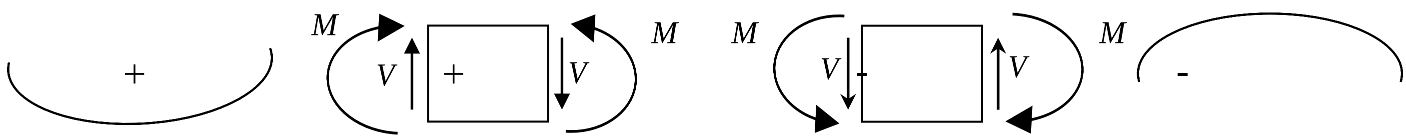
Uniformly varying load

Shearing force and bending moment diagrams show the variation of these quantities along the length of a beam for any fixed loading condition. At every section in a beam carrying transverse loads there will be resultant forces on either side of the section which, for equilibrium, must be equal and opposite.

Shearing force at the section is defined as the algebraic sum of the forces taken on one side of the section. The bending moment is defined as the algebraic sum of the moments of the forces about the section, taken on either side of the section.

Sign Convention:

Forces upwards to the left of a section or downwards to the right of a section are positive. Clockwise moments to the left and counter clockwise to the right are positive.



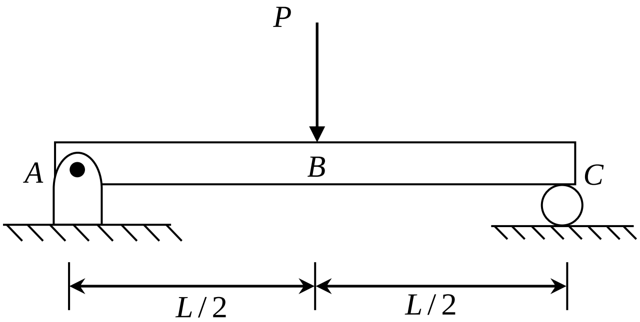
Procedure of Analysis:

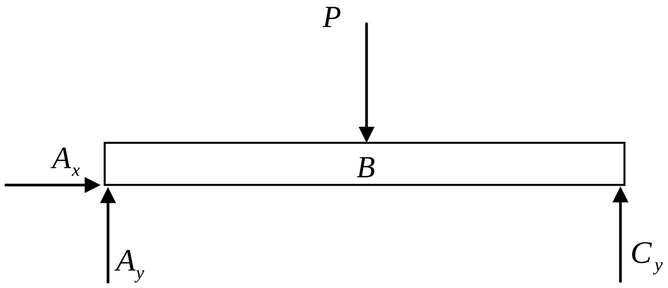
The shear and moment diagrams for a beam can be constructed using the following procedure:-

1. Determine all the reactive forces and couple moments acting on the beam, and resolve all the forces into components acting perpendicular and parallel to the beam's axis.
2. Specify separate coordinates x having an origin at the beam's left end extending to regions of the beam between concentrated forces and/or couple moments, or where there is no discontinuity of distributed loading.
3. Section the beam perpendicular to its axis at each distance x, and draw the free body diagram of one of the segments. Be sure V and M are shown acting in their positive sense, in accordance with the sign convention given as above.
4. The shear is obtained by summing forces perpendicular to the beam's axis.
5. The moment is obtained by summing moment about the sectioned end of the segment.

6. Plot the shear diagram(V versus x) and the moment diagram(M versus x). If numerical values of the functions describing V and M are positive, the values are plotted above the x-axis, whereas negative values are plotted below the axis.

Example 33: Draw the shear and moment diagrams for the beam shown below.





$\sum F_x = 0$

$A_x=0$

$\sum M_C = 0$

$P\times L/2-A_y\times L=0$

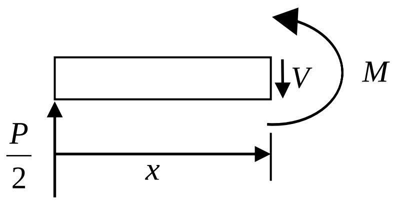
$A_y=P/2$

$\sum F_y = 0$

$C_y+A_y-P=0$

$C_y=P/2$

- Segment AB



$\sum F_y = 0$

$\frac{P}{2}-V=0$

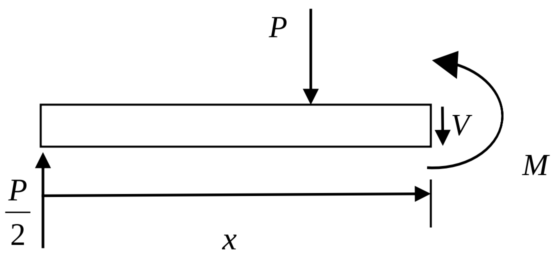
$V=\frac{P}{2}$

$\sum M = 0$

$M-\frac{P}{2}\times x=0$

$M=\frac{P}{2}x$

- Segment BC



$\sum F_y = 0$

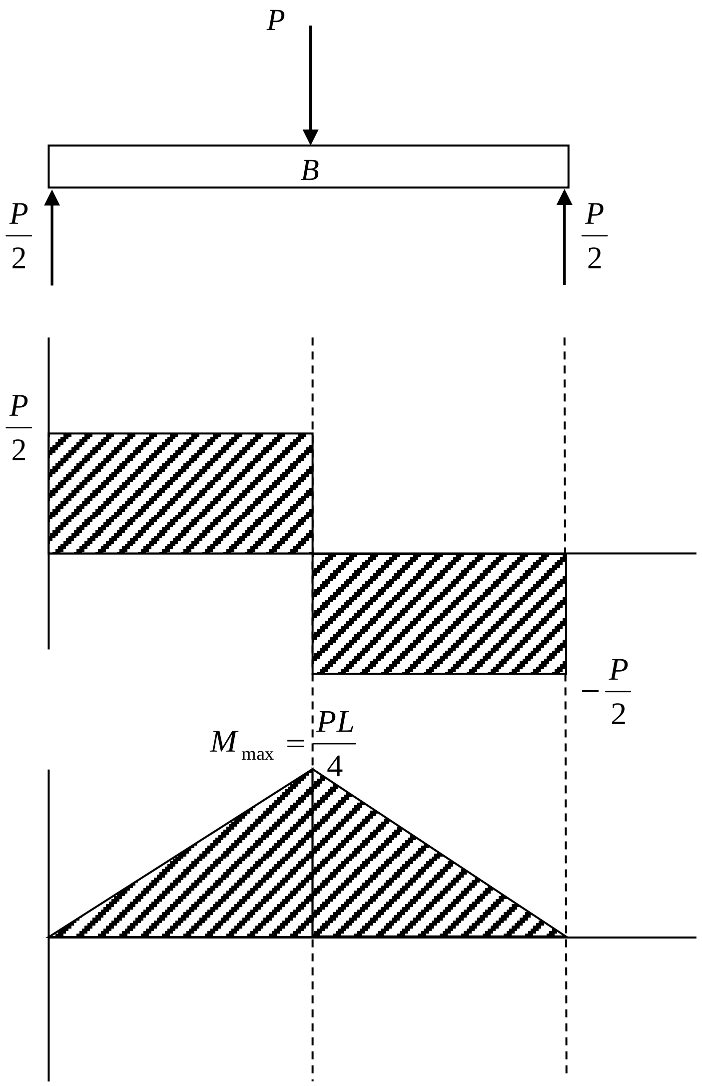
$\frac{P}{2}$-P-V=0

V=-$\frac{P}{2}$

$\sum M = 0$

M-$\frac{P}{2}$×x+P(x-$\frac{L}{2}$)=0

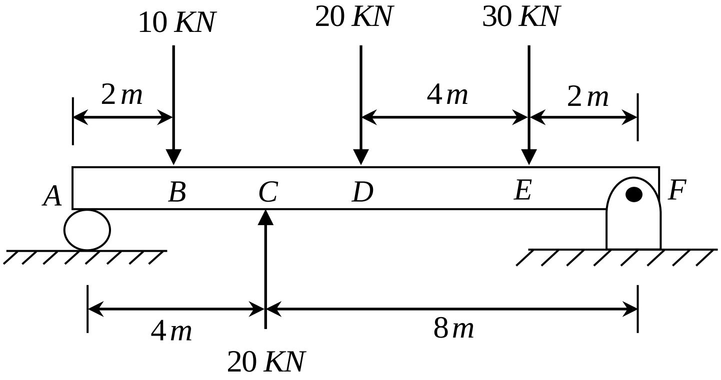
M=$\frac{P}{2}$(L-x)



S.F. diagram

B.M. diagram

Example 34: Draw the shear and moment diagrams for the beam shown below.



$\sum F_x = 0$

$F_x=0$

$\sum M_F = 0$

$-A_y\times12+10\times10-20\times8+20\times6$

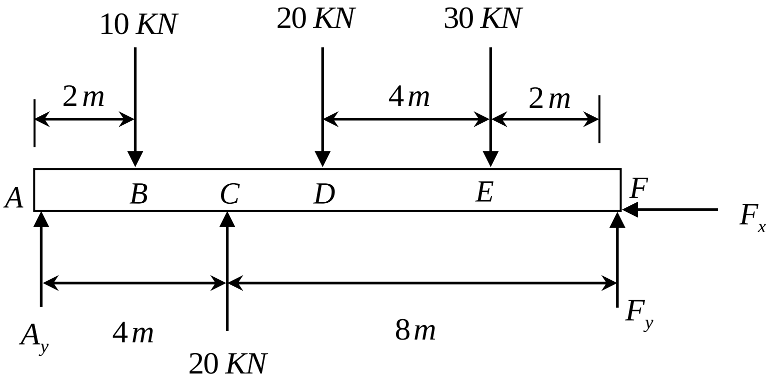
$+30\times2=0$

$A_y=10$ KN

$\sum F_y = 0$

$10-10+20-20-30+F_y=0$

$F_y=30$ KN



- Segment AB $\quad 0 \le x \le 2$

$\sum F_y = 0$

10-V=0

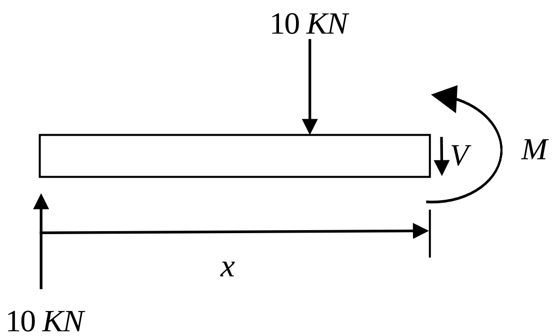
V=10 KN

$\sum M = 0$

M-10×x=0

M=10x

- Segment BC $\quad 2 \le x \le 4$



$\sum F_y = 0$

10-10-V=0

V=0

$\sum M = 0$

M-10x+10(x-2)=0

M=20 KN.m

- Segment CD $\quad 4 \le x \le 6$

$\sum F_y = 0$

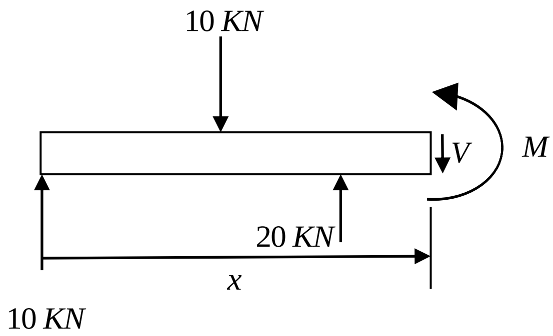
10-10+20-V=0

V=20 KN

$\sum M = 0$

M-10x+10(x-2)-20(x-4)=0

M=20(x-3)



- Segment DE $6 \le x \le 10$

$\sum F_y = 0$

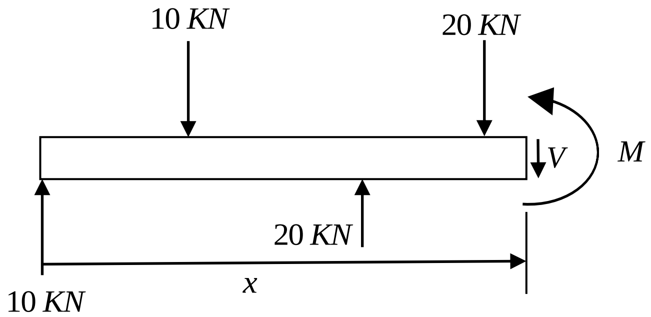
10-10+20-20-V=0

V=0

$\sum M = 0$

M-10x+10(x-2)-20(x-4)+20(x-6)=0

M=60 KN.m



- Segment EF $10 \le x \le 12$

$\sum F_y = 0$

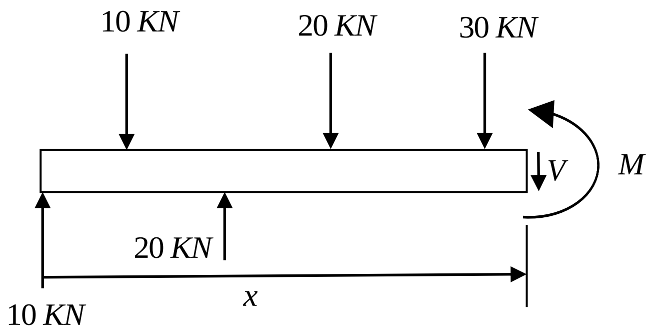
10-10+20-20-30-V=0

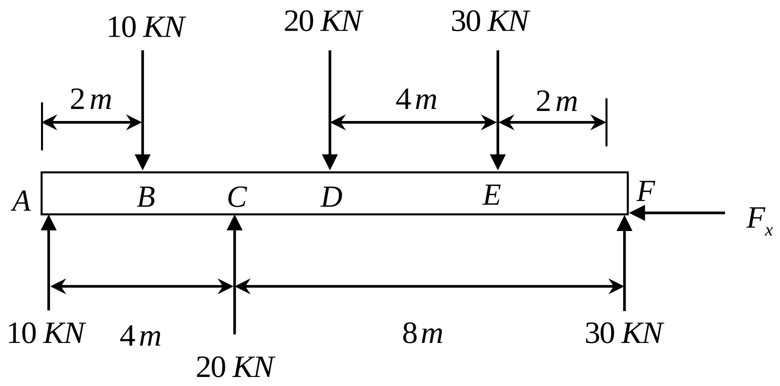
V=-30 KN

$\sum M = 0$

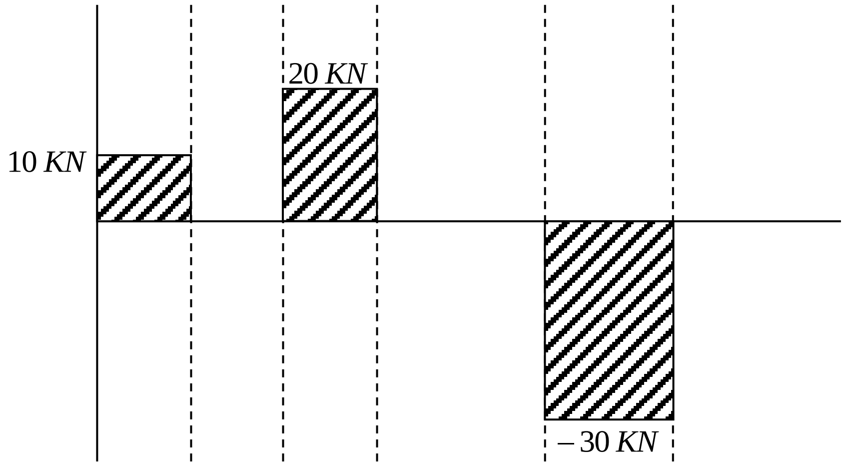
M-10x+10(x-2)-20(x-4)+20(x-6)
+30(x-10)=0

M=30(12-x)

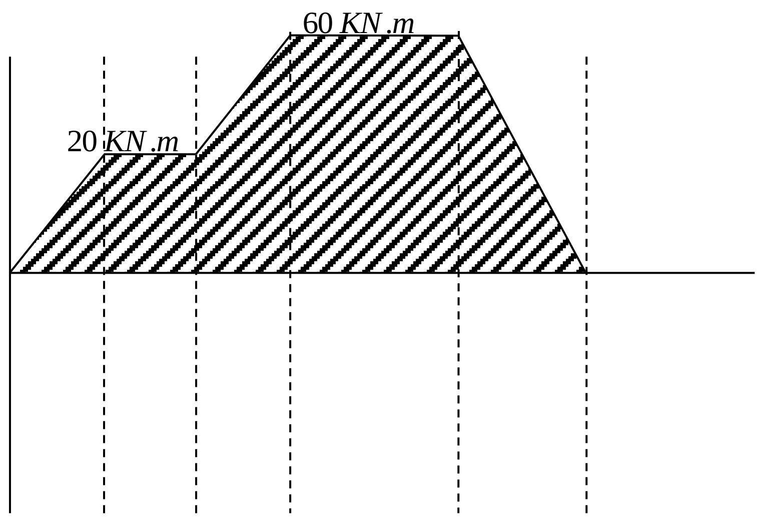




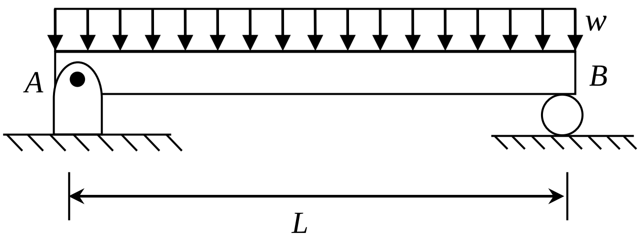
S.F Diagram



B.M. Diagram



Example 35: Draw the shear and moment diagrams for the beam shown below.



$\sum F_x = 0$

$A_x=0$

$\sum M_B = 0$

$wL\frac{L}{2}-A_yL=0$

$A_y=\frac{wL}{2}$

$\sum F_y = 0$

$\frac{wL}{2}+B_y-wL=0$

$B_y=\frac{wL}{2}$

$\sum F_y = 0$

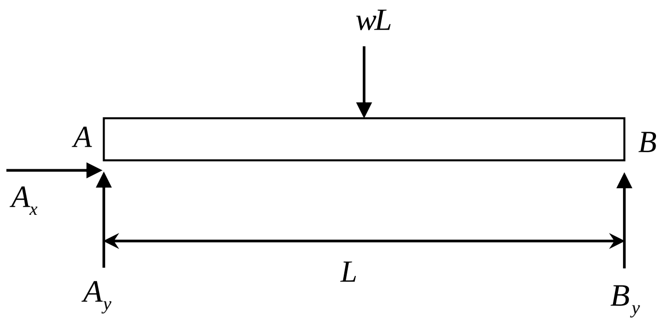
$\frac{wL}{2}-wx-V=0$

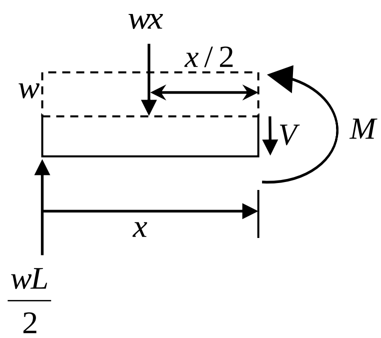
$V=-w(x-\frac{L}{2})$

$\sum M = 0$

$M-\frac{wL}{2}x+wx(\frac{x}{2})=0$

$M=\frac{w}{2}(xL-x^2)$





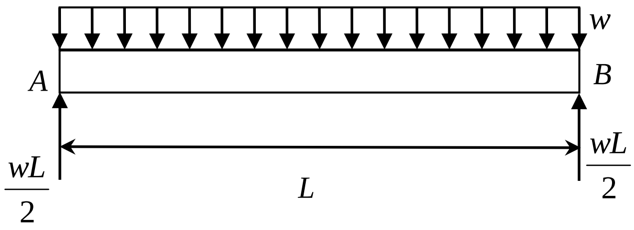
Maximum moment occur when $\frac{dM}{dx} = 0$

$\frac{dM}{dx} = \frac{w}{2}(L-2x) = 0$

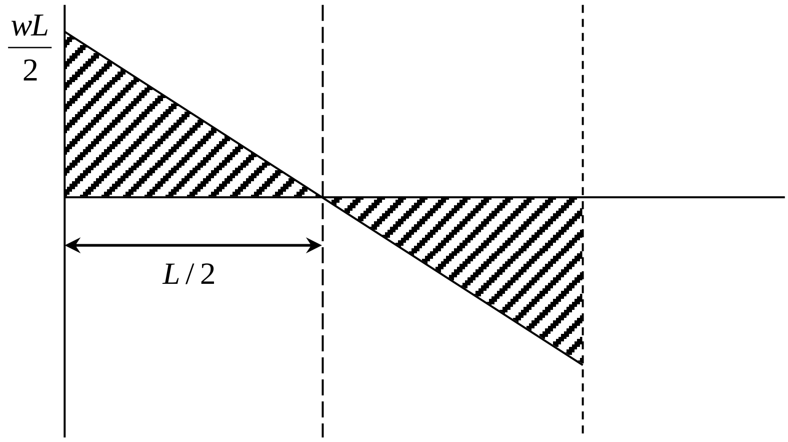
$L-2x=0$

$x=\frac{L}{2}$

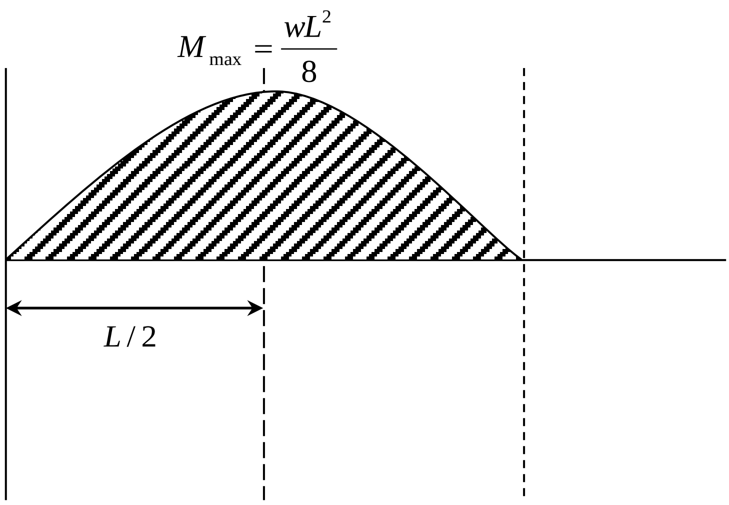
Location of maximum moment



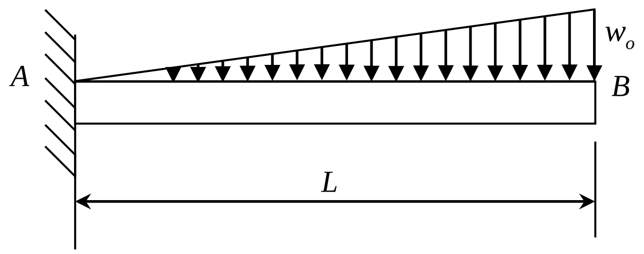
S.F Diagram

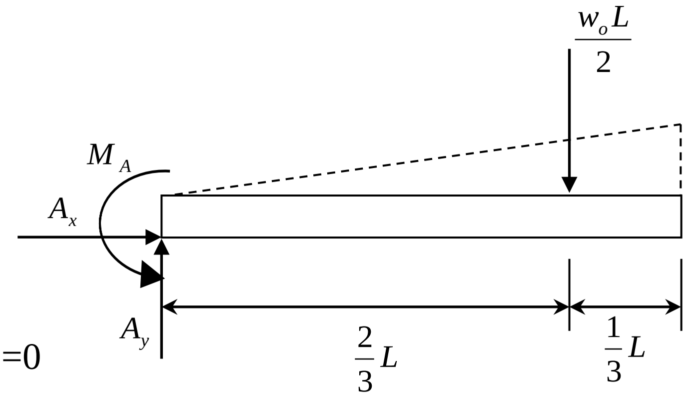


B.M Diagram



Example 36: Draw the shear and moment diagrams for the beam shown below.





$\sum F_x = 0$

$A_x=0$

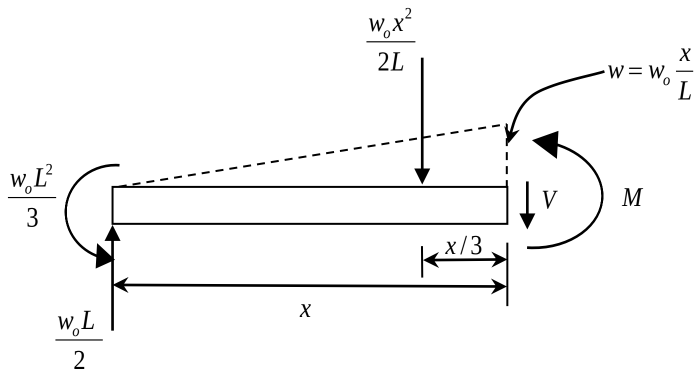
$\sum F_y = 0$

$A_y=\frac{w_oL}{2}$

$\sum M_A = 0$

$M_A-\frac{w_oL}{2}\,\frac{2}{3}L=0$

$M_A=\frac{w_oL^2}{3}$



$\sum F_y = 0$

$\frac{w_oL}{2}-\frac{w_ox^2}{2L}-V=0$

$V=\frac{w_o}{2}(L-\frac{x^2}{L})$

$V_{max}=\frac{w_oL}{2}$

Maximum shear force occur at $\frac{dV}{dx}=0$

$\frac{dV}{dx}=-\frac{w_ox}{L}=0$

x=0

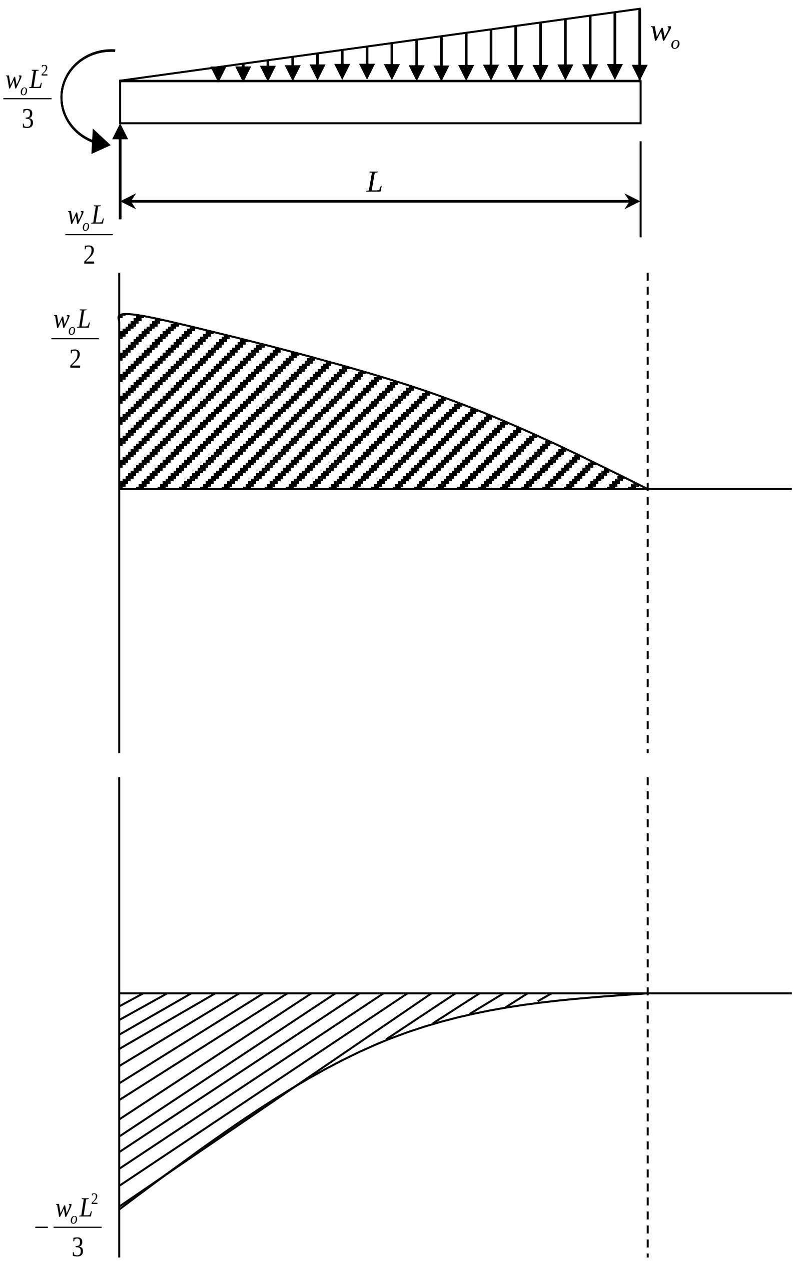
$\sum M = 0$

$M+\frac{w_oL^2}{3}-\frac{w_oL}{2}x+\frac{w_ox^2}{2L}\,\frac{1}{3}x=0$

$$M=\frac{w_o}{6L}(3L^2x-x^3-2L^3)$$

$$M_{max}= -\frac{w_oL^2}{3}$$

S.F Diagram



Example 37: The horizontal beam ***AD*** is loaded by a uniform distributed load of ***5 KN per meter*** of length and is also subjected to the concentrated force of ***10 KN*** applied as shown below. Determine the shearing force and bending moment diagrams.

$\sum F_x = 0$

$A_x=0$

$\sum M_A = 0$

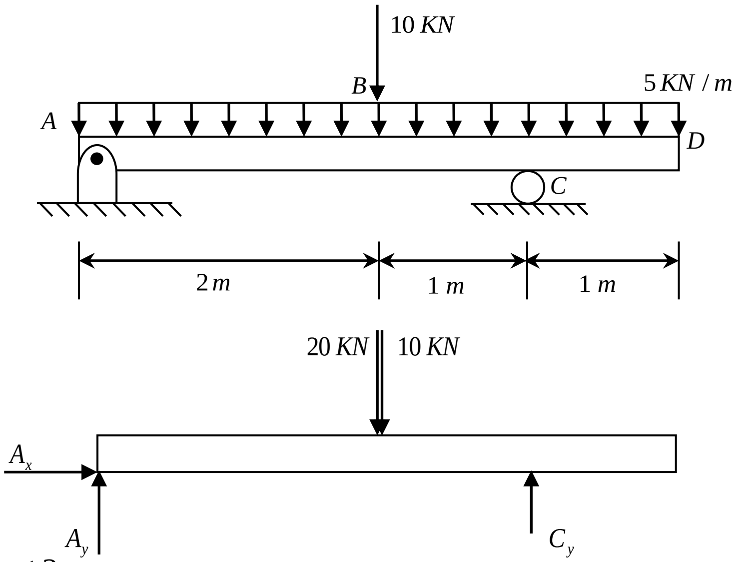
$C_y \times 3 - 30 \times 2 = 0$

$C_y = 20$ KN

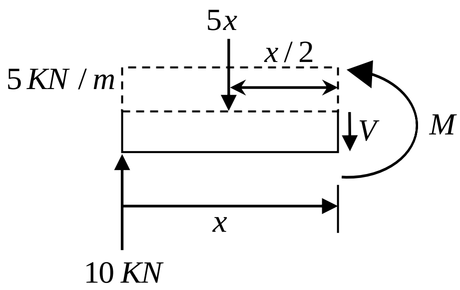
$\sum F_y = 0$

$A_y + 20 - 30 = 0$

$A_y = 10$ KN



- Segment AB $\quad 0 \le x \le 2$



$\sum F_y = 0$

$10 - 5x - V = 0$

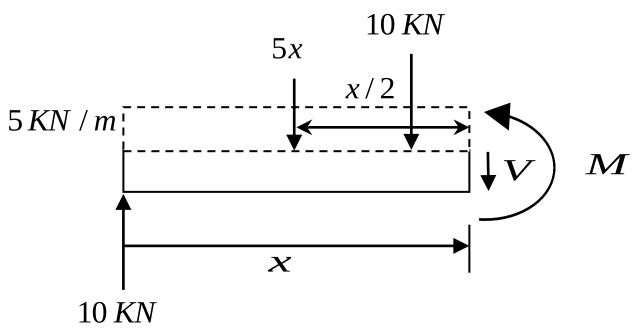
$V = 10 - 5x$

$\sum M = 0$

$M - 10x + 5x\frac{x}{2} = 0$

$M = 5x(2 - \frac{x}{2})$

- Segment BC $\quad 2 \le x \le 3$



$\sum F_y = 0$

$10 - 5x - 10 - V = 0$

$V = -5x$

$\sum M = 0$

$M - 10x + 5x\frac{x}{2} + 10(x-2) = 0$

$M = 20 - \frac{5}{2}x^2$

- Segment CD $\quad 3 \le x \le 4$

$\sum F_y = 0$

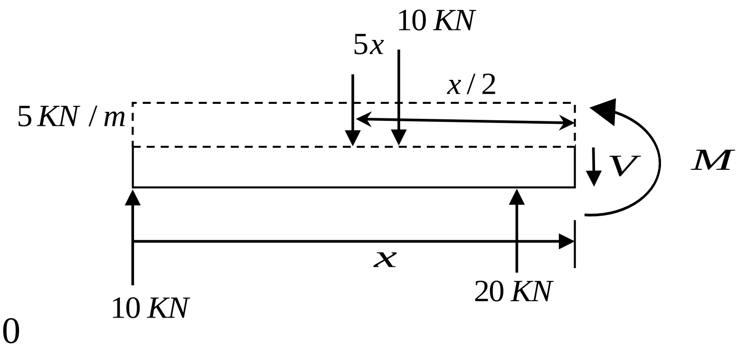
10-5x-10+20-V=0

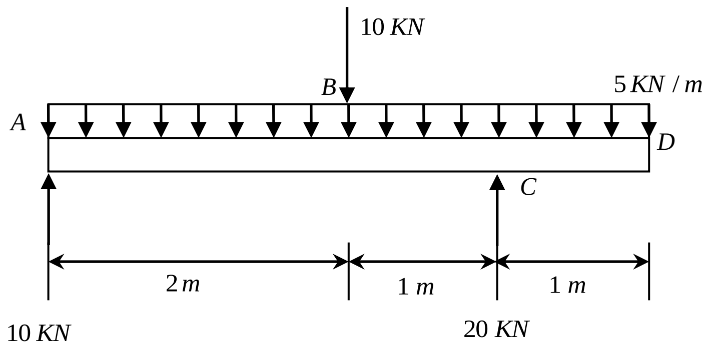
V=20-5x

$\sum M = 0$

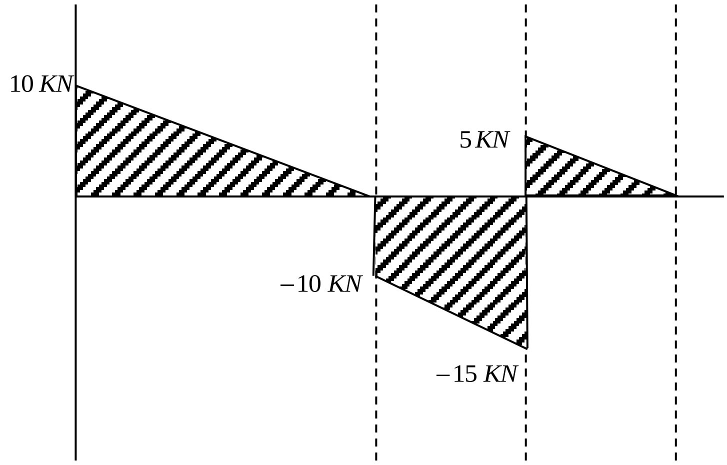
M-10x+5x$\frac{x}{2}$+10(x-2)-20(x-3)=0

M=-40+20x-$\frac{5}{2}$x^2

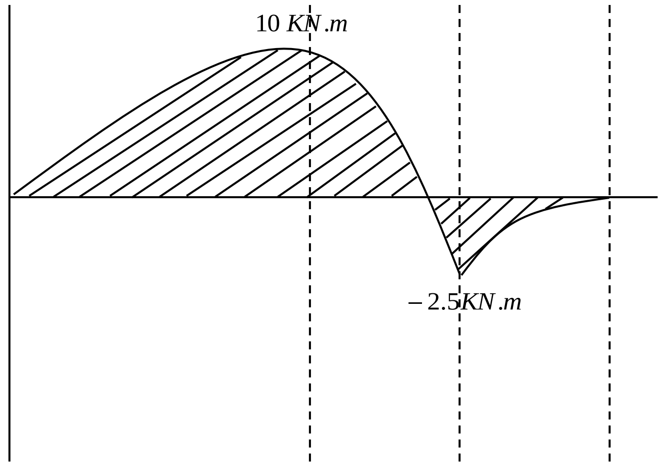




S.F Diagram



B.M Diagram



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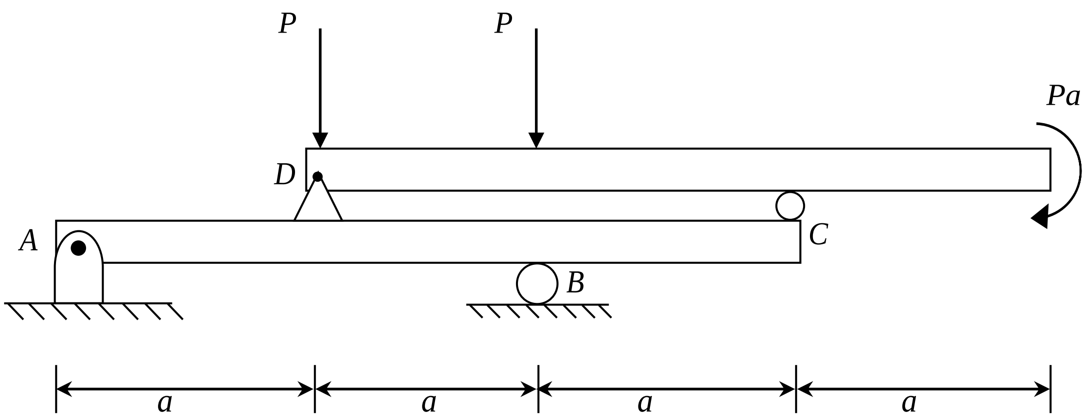
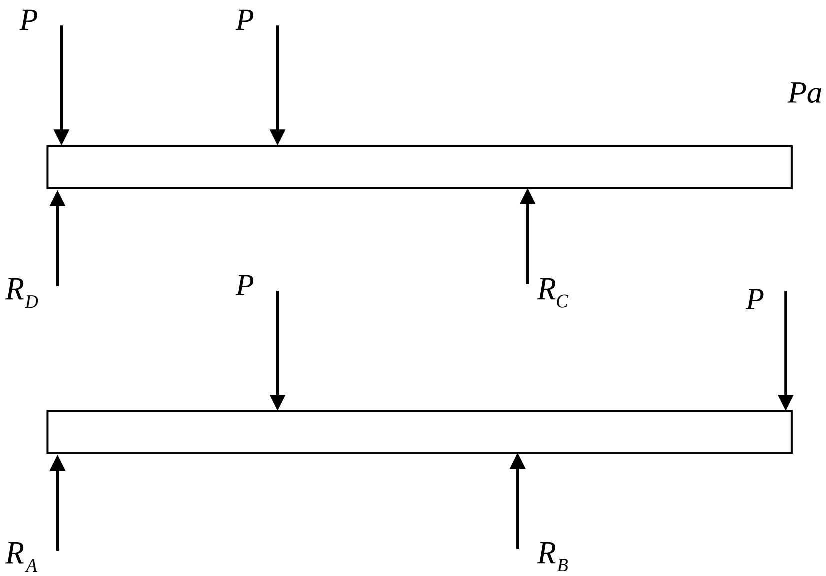
Example 38: A beam ***ABC*** is simply supported at ***A*** and ***B*** and has an overhang ***BC***. The beam is loaded by two forces ***P*** and a clockwise couple of moment ***Pa*** that act through the arrangement shown. Draw the shear force and bending moment diagrams for beam ***ABC***.





$\sum M_D = 0$

-Pa+R_C(2a)-Pa=0

R_C=P

$\sum F_y = 0$

R_D+P-P-P=0

R_D=P

$\sum M_A = 0$

R_B(2a)-Pa-P(3a)=0

R_B=2P

$\sum F_y = 0$

R_A+2P-P-P=0

R_A=0

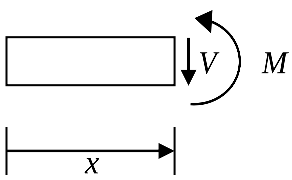
- Segment AD $0 \le x \le a$

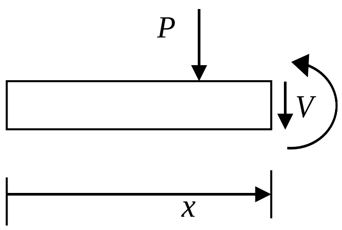
$\sum F_y = 0$

V=0

$\sum M = 0$

M=0





- Segment DB $a \le x \le 2a$

$\sum F_y = 0$

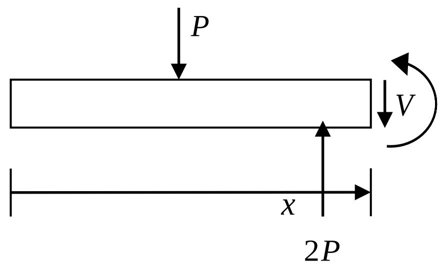
-V-P=0

V=-P

$\sum M = 0$

M+P(x-a)=0

M=P(a-x)



- Segment DB $2a \le x \le 3a$

$\sum F_y = 0$

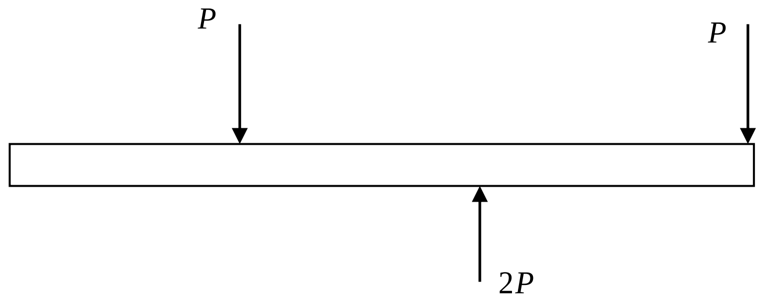
2P-P-V=0

V=P

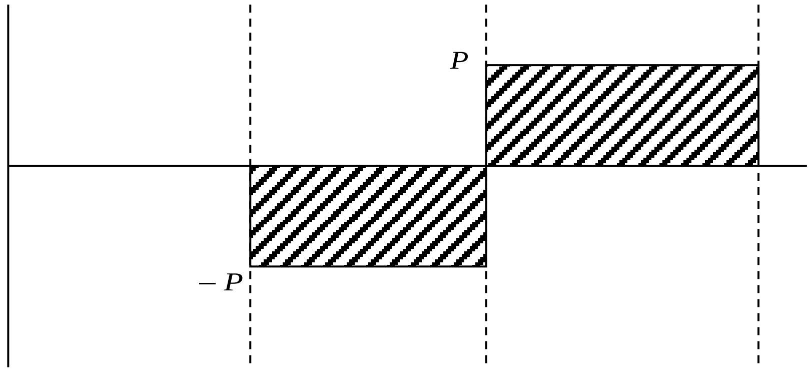
$\sum M = 0$

M+P(x-a)-2P(x-2a)=0

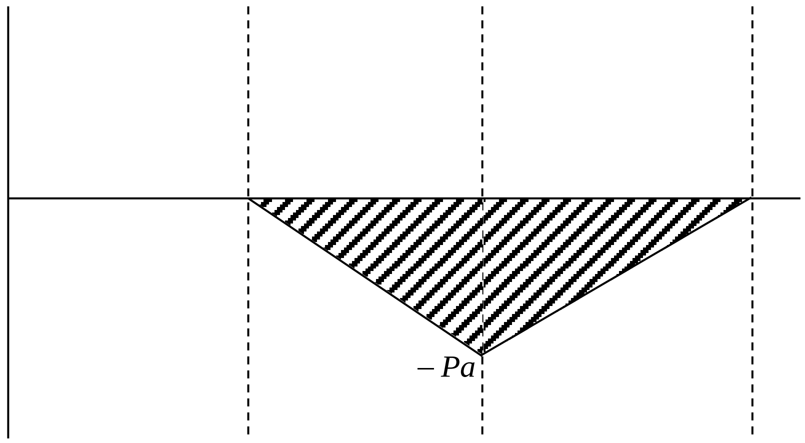
M=P(x-3a)



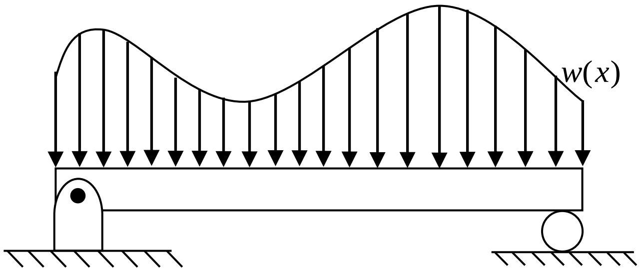
S.F. Diagram



B.M. Diagram



Graphical Method for Constructing Shear and Moment Diagram:



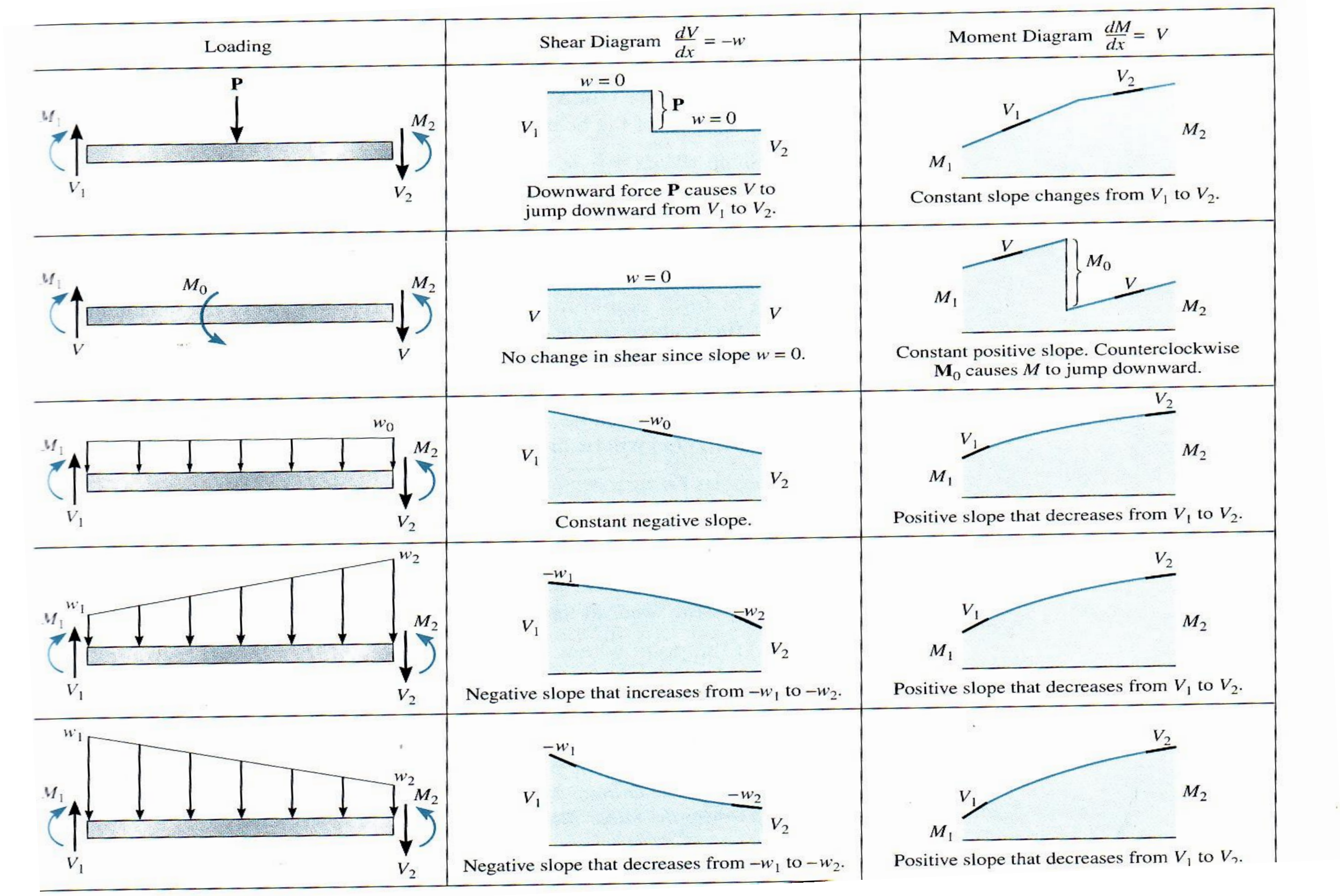
$$\frac{dV}{dx} = -w(x)$$

Slope of shear diagram at each point=-distributed load intensity at each point.

$$\frac{dM}{dx} = V$$

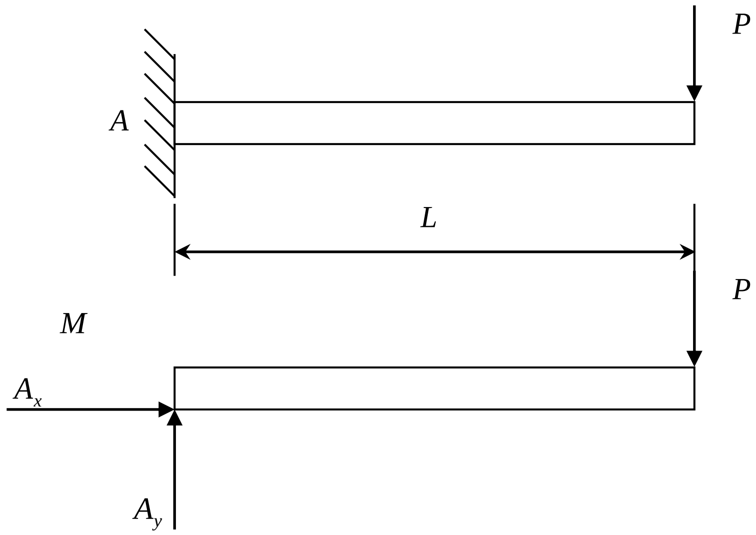
Slope of moment diagram at each point=shear at each point.

- When the force acts downward on the beam, ΔV is negative so the shear will jump downward. Likewise, if the force acts upward, the jump will be upward.
- If moment M_o is applied clockwise on the beam, ΔM is positive so the moment diagram will jump upward. Likewise, when M_o acts counterclockwise, the jump will be downward.

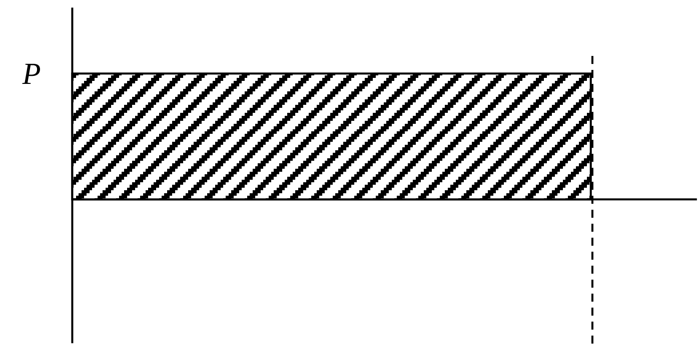


Example 39: Draw the shear and moment diagrams for the beam shown below.

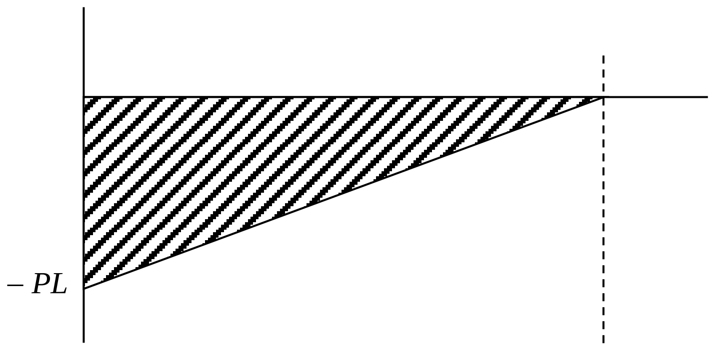
$\sum F_x = 0$
$A_x=0$
$\sum F_y = 0$
$A_y-P=0$
$A_y=P$
$\sum M_A = 0$
$M-PL=0$
$M=PL$



At x=0 V=P
At x=L V=P



At x=0 M=-PL
At x=L M=0



Example 40: Draw the shear and moment diagrams for the beam shown below.

$\sum M_A = 0$

$B_y \times 5.5 - 10 - 60 \times 2 = 0$

$B_y = 23.63$ KN

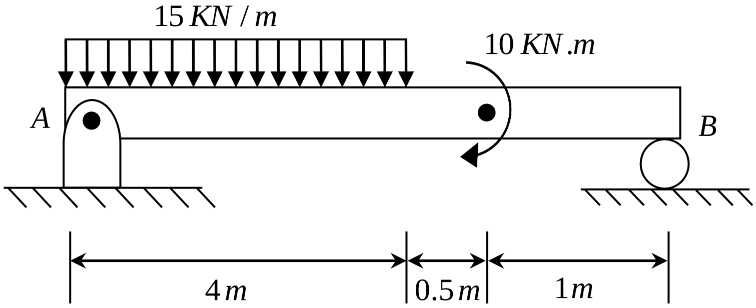
$\sum F_y = 0$

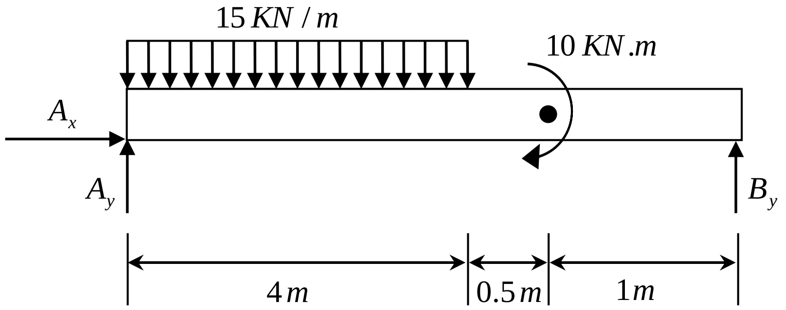
$23.63 + A_y - 60 = 0$

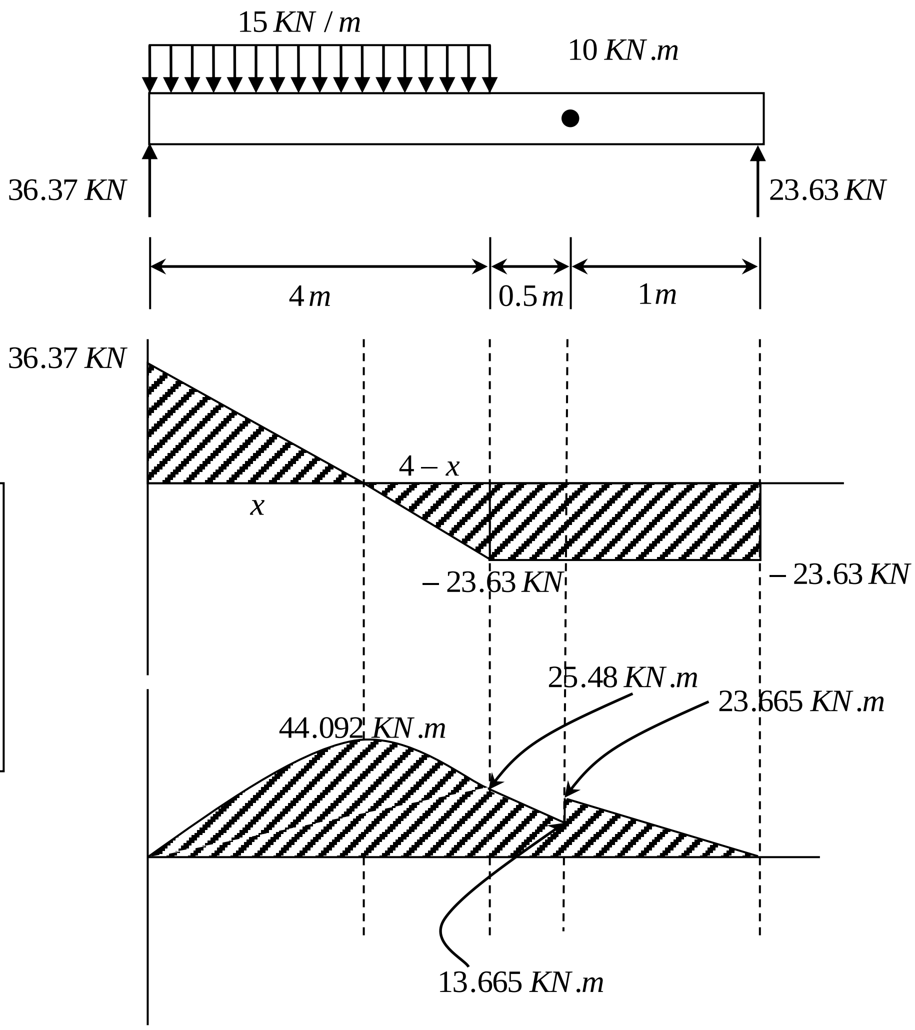
$A_y = 36.37$ Kn

$\sum F_x = 0$

$A_x = 0$







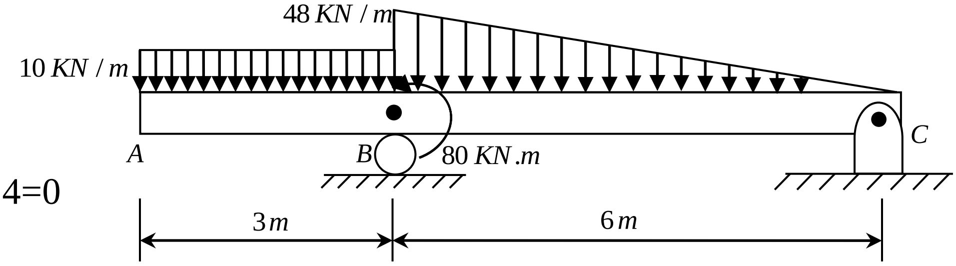
$$\frac{x}{36.37} = \frac{4-x}{23.63}$$

$23.63x = 36.37 \times 4 - 36.37x$

$x = 2.4246$ m

Maximum bending moment occur when V=0, at x=2.4246 m

Example 41: Draw the shear and moment diagrams for the beam shown below.



$\sum M_C = 0$

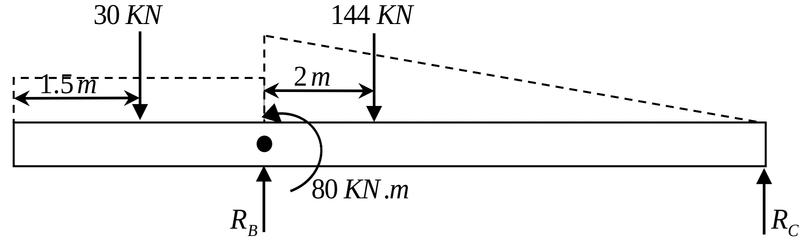
$80 - R_B \times 6 + 30 \times 7.5 + 144 \times 4 = 0$

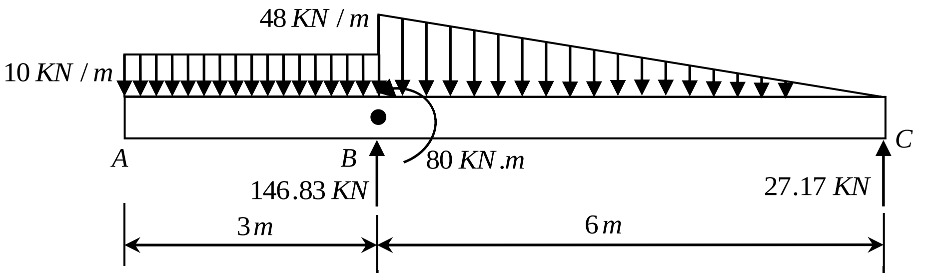
$R_B = 146.83$ KN

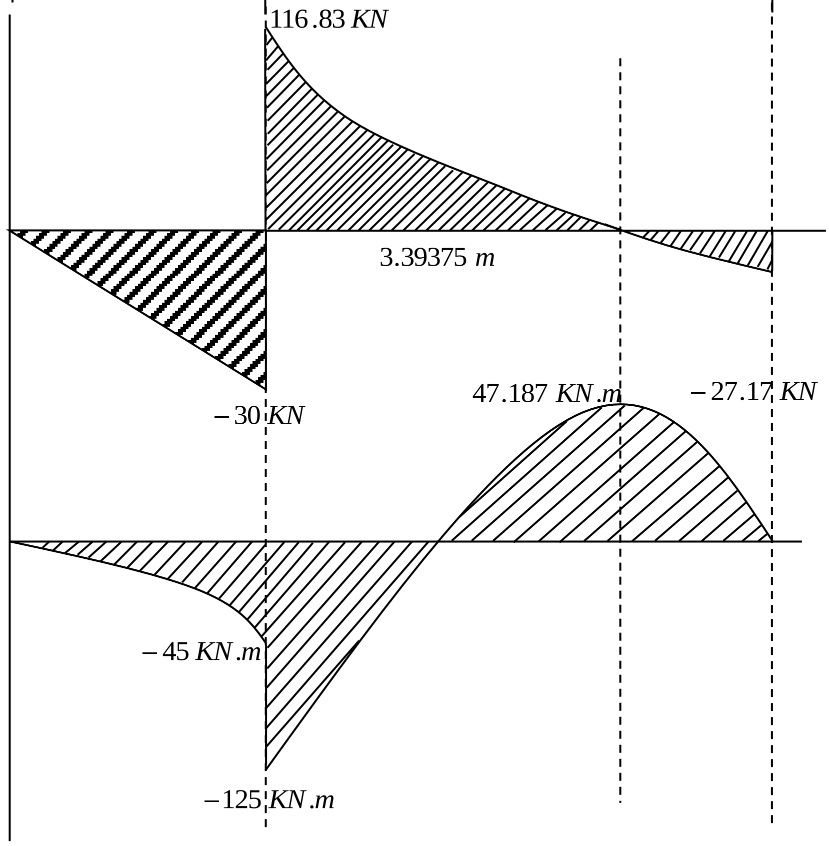
$\sum F_y = 0$

$R_C + 146.83 - 30 - 144 = 0$

$R_C = 27.17$ KN







For segment $3 \le x \le 6$

$V = 116.83 + 4(x-3)^2 - 48(x-3)$

Maximum bending moment occur when $V = 0$

$0 = 116.83 + 4(x-3)^2 - 48(x-3)$

$x^2 - 18x + 74.2075 = 0$

$x = 6.39375$ m from left end

Example 42: Draw the shear and moment diagrams for the beam shown below.

$\sum F_x = 0$

$A_x=0$

$\sum M_A = 0$

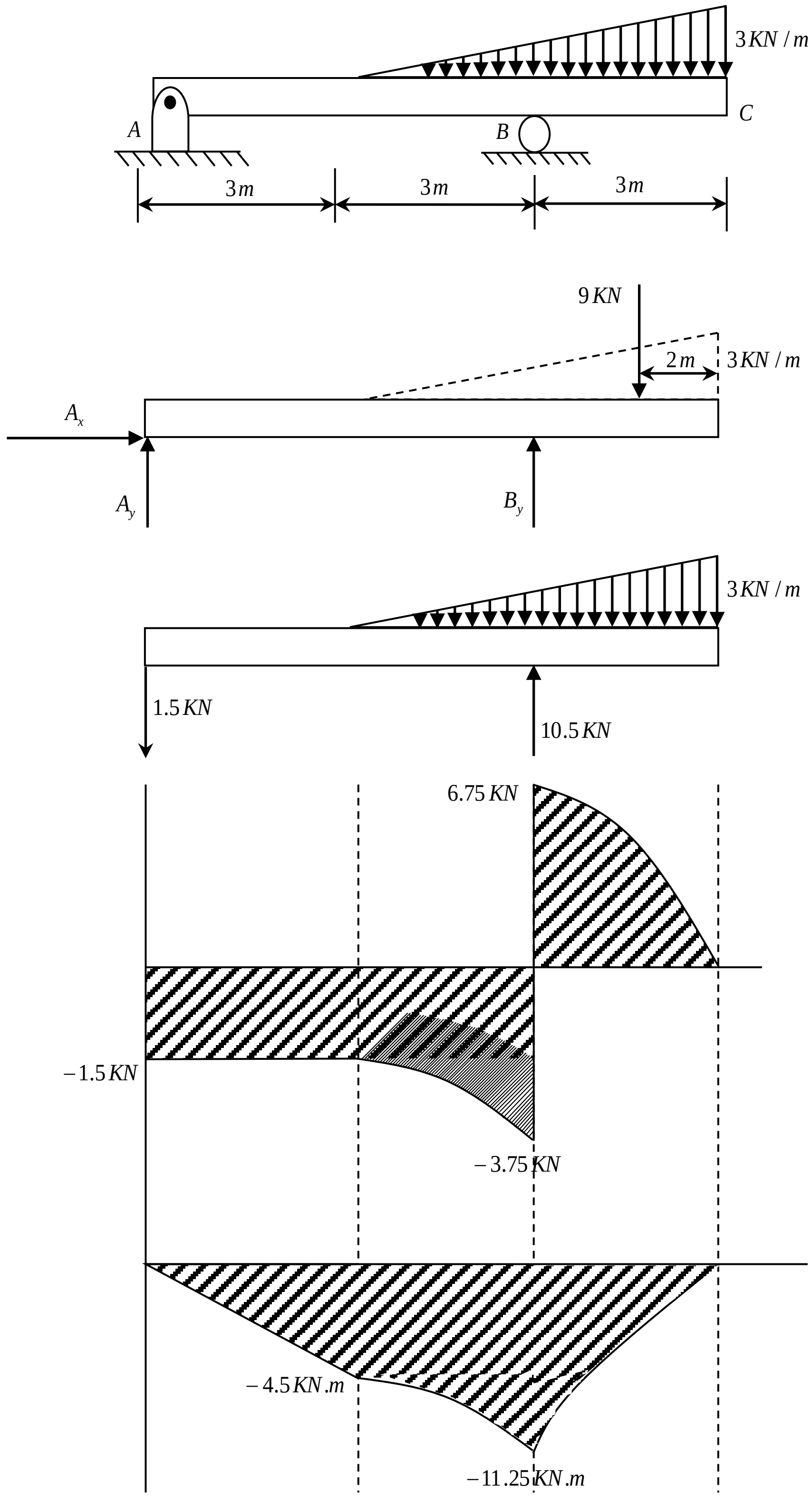
$B_y \times 6 - 9 \times 7 = 0$

$B_y = 10.5$ KN

$\sum F_y = 0$

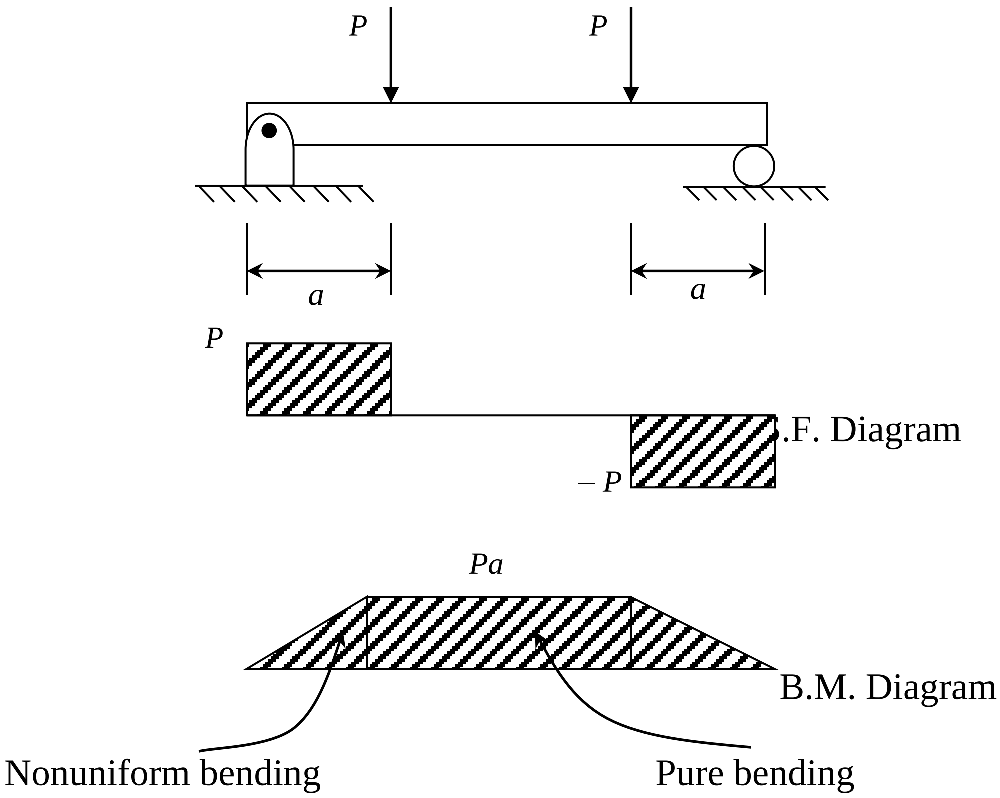
$A_y + 10.5 - 9 = 0$

$A_y = -1.5$ KN



Stresses in Beams:

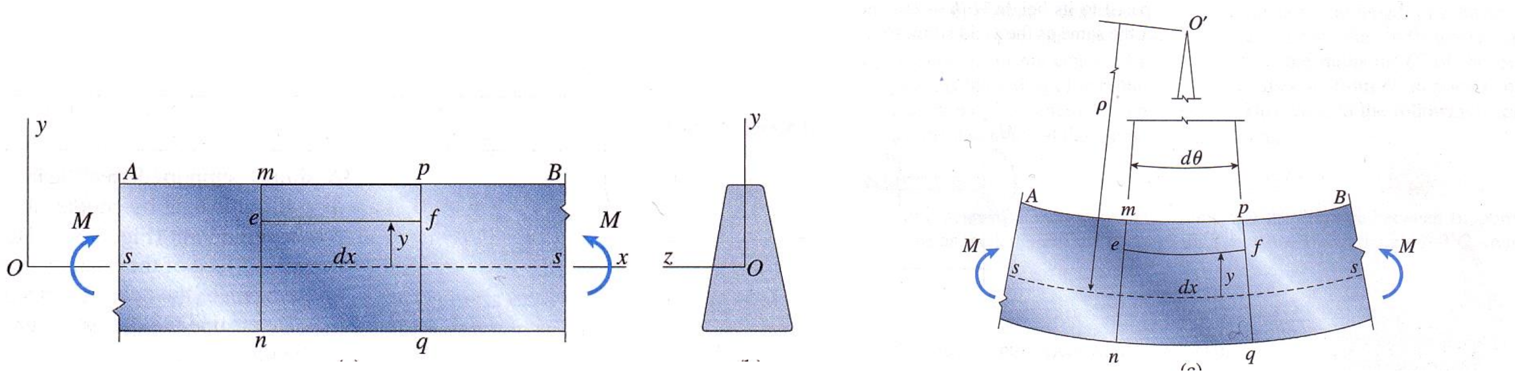
Pure bending refers to flexure of a beam under a constant bending moment. Therefore, pure bending occurs only in regions of a beam where the shear force is zero. Nonuniform bending refers to flexure in the presence of shear forces, which means that the bending moment changes as we move along the axis of the beam.



Assumptions:

1. The beam is initially straight and unstressed.
2. The material of the beam is perfectly homogeneous.
3. The elastic limit is nowhere exceeded.
4. Young's modulus for the material is the same in tension and compression.
5. Plane cross-sections remain plane before and after bending.
6. Every cross-section of the beam is symmetrical about the plane of bending i.e. about an axis perpendicular to the N.A.
7. There is no resultant force perpendicular to any cross-section.

If we now considered a beam initially unstressed and subjected to a constant bending moment along its length, i.e. pure bending as would be obtained by applying equal couples at each end, it will bend to a radius ρ as shown below.



As a result of this bending the top fibers of the beam will be subjected to compression and the bottom to tension. Its reasonable to suppose, that somewhere between the two there are points at which the stress is zero, these points is termed the neutral axis. The neutral axis will always pass through the centre of area or centroid.

The length ***L₁*** of the line ***ef*** after bending takes place is:

$L_1=(\rho-y)d\theta$

$d\theta=\dfrac{dx}{\rho}$

$L_1=(1-\dfrac{y}{\rho})dx$

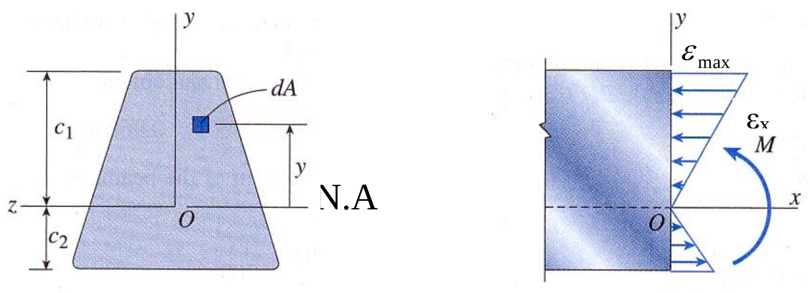
The original length of line ***ef*** is dx

$$\text{Strain}(\varepsilon_x)=\frac{L_1-originallength}{originallength}=\frac{(1-\frac{y}{\rho})dx-dx}{dx}=-\frac{y}{\rho}$$

$\varepsilon_x=-ky$

where k is the curvature.

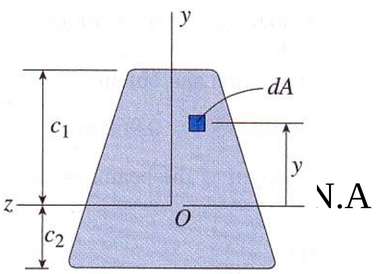
The longitudinal normal strain will vary linearly with y from the neutral axis. A contraction ($-\varepsilon_x$) will occur in fibers located above the neutral axis (+y), whereas elongation ($+\varepsilon_x$) will occur in fibers located below the neutral axis (-y).

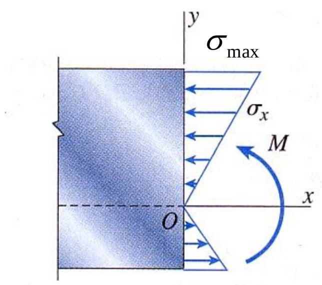


$\varepsilon_x=-(\dfrac{y}{c_1})\ \varepsilon_{max}$

By using Hook's law $\sigma_x=E\varepsilon_x$

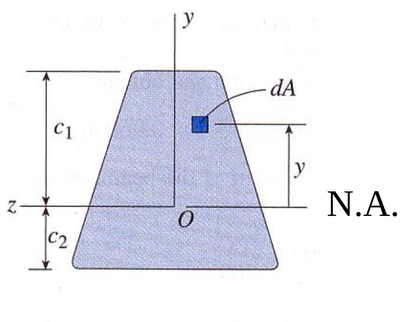
$\sigma_x=-Eky=-\dfrac{E}{\rho}y$





$\sigma_x = -\left(\frac{y}{c_1}\right)\sigma_{max}$

Normal stress will vary linearly with y from the neutral axis. Stress will vary from zero at the neutral axis to a maximum value σ_{max} a distance c_1 farthest from neutral axis.



$dF = \sigma_x dA$

$M = \int_A y\,dF = \int_A (\sigma_x dA) y$

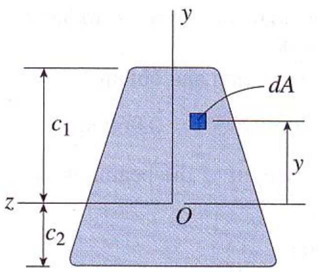
$= \int_A \left(-\frac{y}{c_1}\sigma_{max}\right) y\,dA$

$M = \frac{\sigma_{max}}{c_1}\int_A y^2 dA$

$\int_A y^2 dA = I$ moment of inertia

$\sigma_{max} = \frac{Mc_1}{I}$

σ_{max}: The maximum normal stress in the member, which occurs at a point on the cross sectional area farthest away from the neutral axis.
M: The resultant internal moment.
I: The moment of inertia of the cross sectional area computed about the neutral axis.
c_1: The perpendicular distance from the neutral axis to a point farthest away from the neutral axis, where σ_{max} acts.



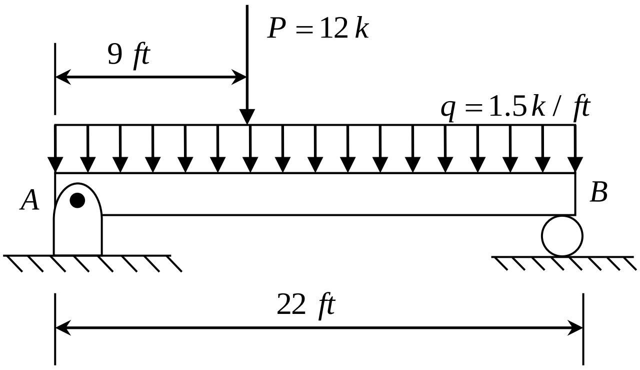
$$\sigma_1 = -\frac{Mc_1}{I} \quad , \sigma_2 = \frac{Mc_2}{I}$$

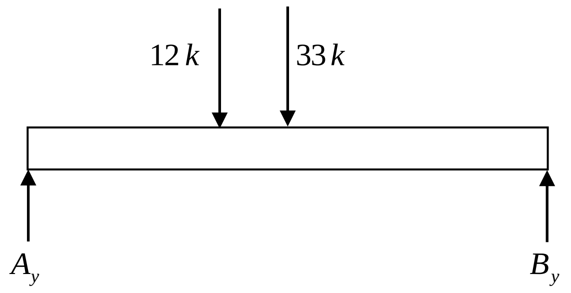
$$\sigma_1 = -\frac{M}{S_1} \quad , \sigma_2 = \frac{M}{S_2}$$

$$S_1 = \frac{I}{c_1} \quad , S_2 = \frac{I}{c_2}$$

The quantities S_1 and S_2 are known as the section moduli of the cross sectional area.

Example 43: A simple beam ***AB*** of span length ***L=22 ft*** supports a uniform load of intensity ***q=1.5 k/ft*** and a concentrated load ***P=12 k***. The uniform load includes an allowance for the weight of the beam. The concentrated load acts at a point ***9 ft*** from the left hand end of the beam. The beam is constructed of glued laminated wood and has a cross section of width ***b=8.75 in*** and height ***h=27 in***. Determine the maximum tensile and compressive stresses in the beam due to bending.





$\sum M_A = 0$

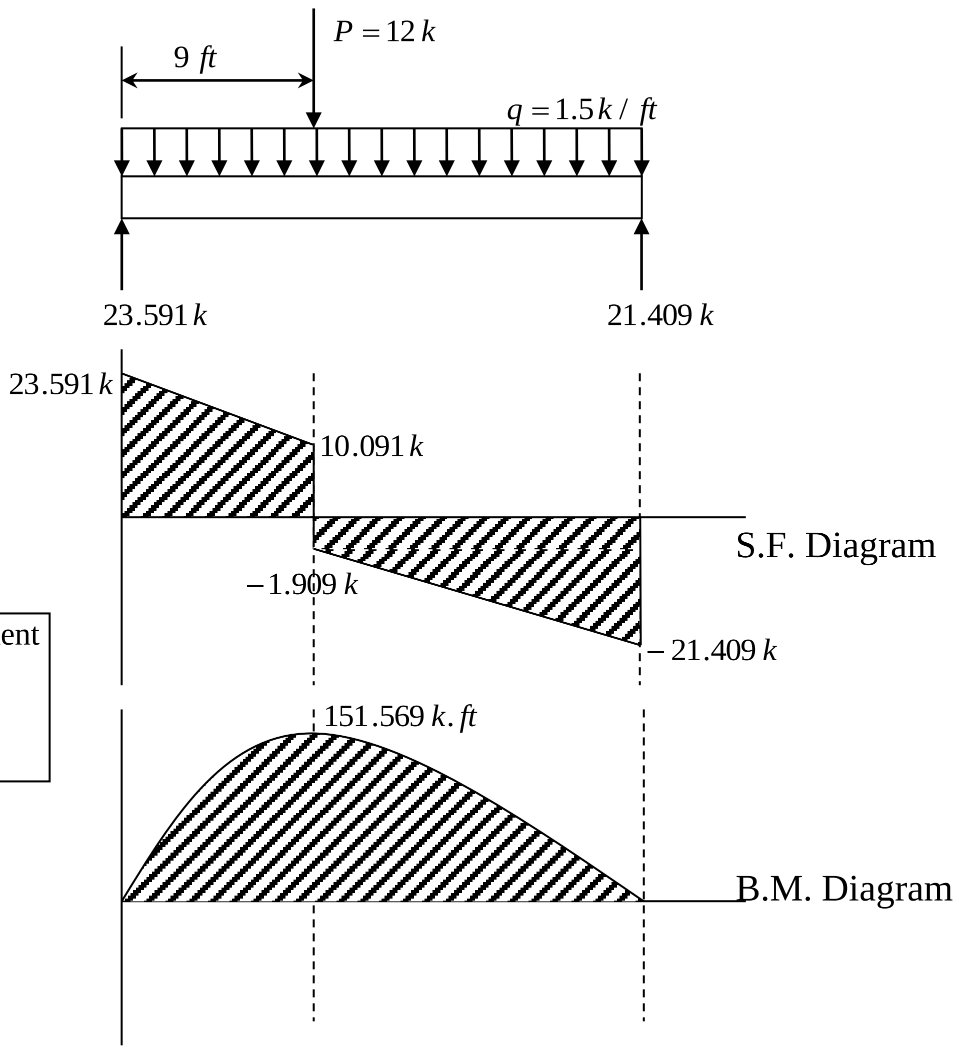
$B_y \times 22 - 12 \times 9 - 33 \times 11 = 0$

$B_y = 21.409$ k

$\sum F_y = 0$

$A_y + 21.409 - 12 - 33 = 0$

$A_y = 23.591$ k



Maximum bending moment
M_{max}=151.569 k.ft
=151.569×12
=1818.828 ksi

$c_1=c_2=13.5$ in

$\sigma_1=-\dfrac{Mc_1}{I}$

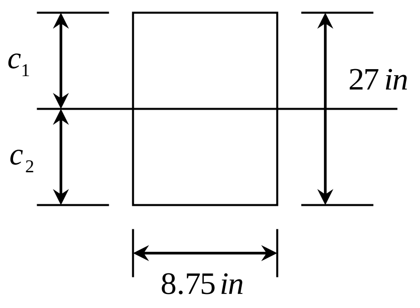
$I=\dfrac{bh^3}{12}=\dfrac{8.75\times(27)^3}{12}=14352.1875\,in^4$

$\sigma_1=-\dfrac{1818.828\times10^3\times13.5}{14352.1875}$

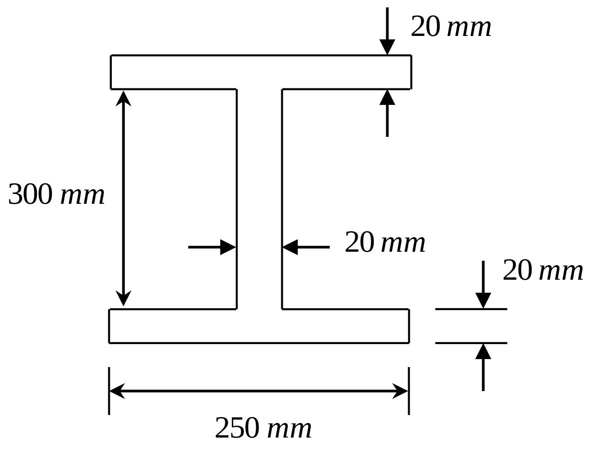
$=-1710.8317$ psi

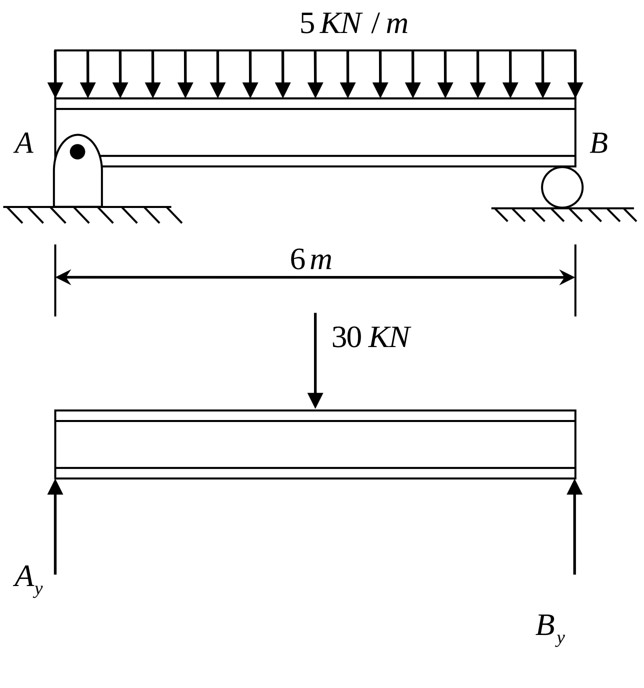
$\sigma_1=\dfrac{1818.828\times10^3\times13.5}{14352.1875}$

$=1710.8317$ psi



Example 44: The simply supported beam has the cross sectional area shown below. Determine the absolute maximum bending stress in the beam and draw the stress distribution over the cross section at this location.





$\sum M_A = 0$

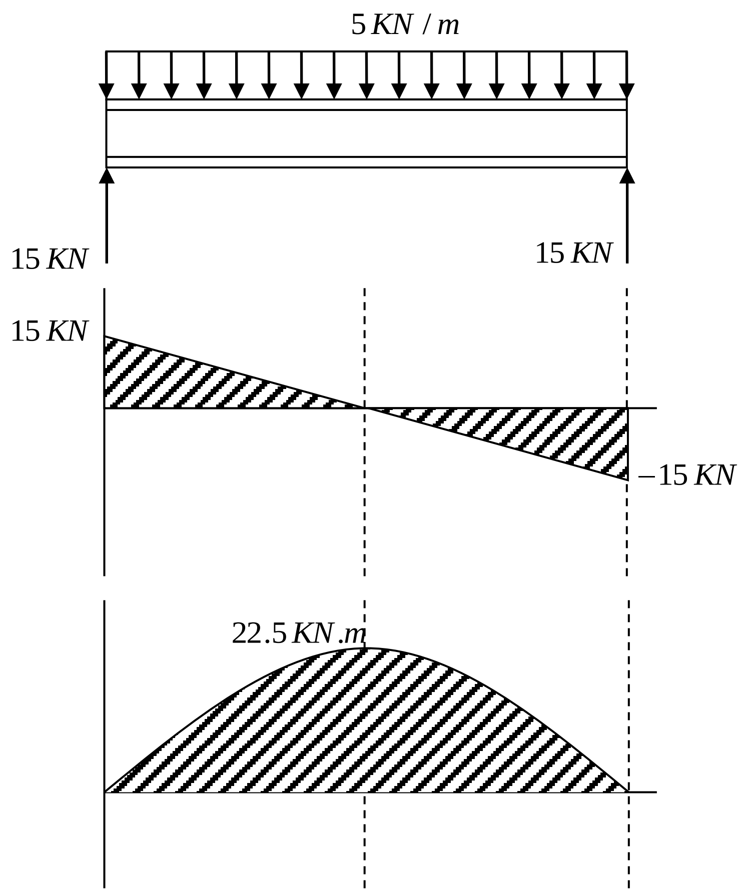
$B_y \times 6 - 30 \times 3 = 0$

$B_y = 15$ KN

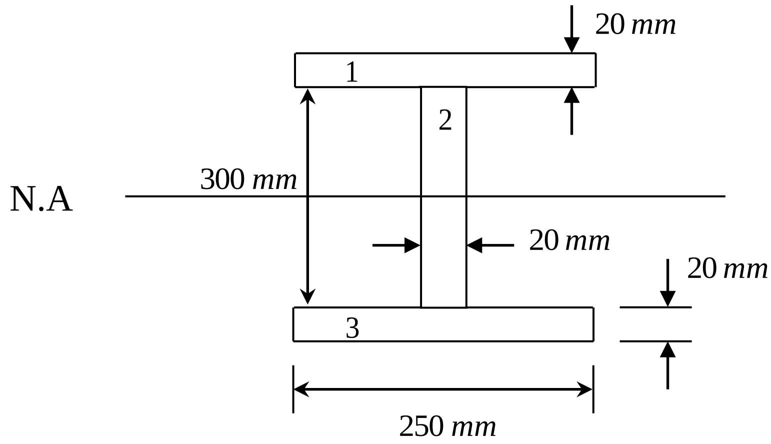
$\sum F_y = 0$

$A_y + 15 - 30 = 0$

$A_y = 15$ KN



Maximum bending moment
$M_{max} = 22.5$ KN.m



$c_1 = c_2 = 170$ mm

$I_1 = \frac{bh^3}{12} + Ad^2$

$I_1 = \frac{250 \times 10^{-3} \times (20 \times 10^{-3})^3}{12} + (250 \times 10^{-3} \times 20 \times 10^{-3}) \times (160 \times 10^{-3})^2$

$I_1 = 128.16667 \times 10^{-6}$ m^4

$I_3 = I_1 = 128.16667 \times 10^{-6}$ m^4

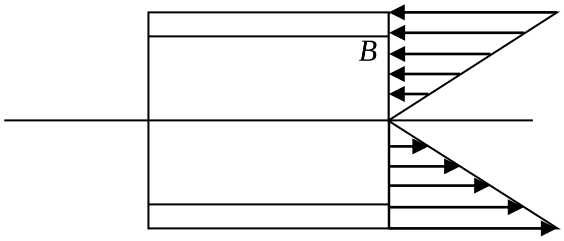
$I_2 = \frac{bh^3}{12} = \frac{(20 \times 10^{-3}) \times (300 \times 10^{-3})^3}{12} = 45 \times 10^{-6}$ m^4

$I = I_1 + I_2 + I_3 = 128.16667 \times 10^{-6} + 128.16667 \times 10^{-6} + 45 \times 10^{-6}$

$I = 301.333 \times 10^{-6}$ m^4

$\sigma_{max} = \frac{Mc_1}{I} = \frac{22.5 \times 10^3 \times 170 \times 10^{-3}}{301.333 \times 10^{-6}}$

$= 12.693598$ MPa.

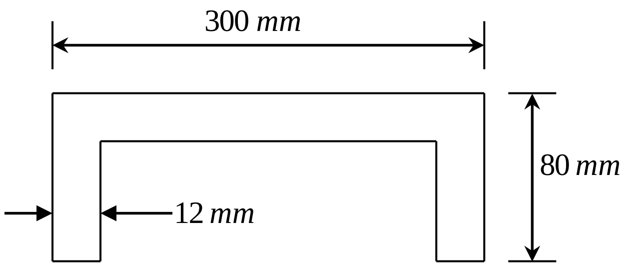


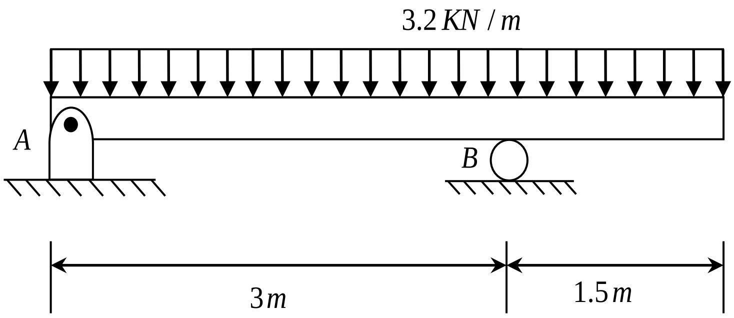
$$\sigma_B = \frac{My_B}{I}$$

$$= \frac{22.5\times10^{3}\times150\times10^{-3}}{301.333\times10^{-6}}$$

$$= 11.200233 \text{ MPa.}$$

Example 45: The beam shown below has a cross section of channel shape with width ***b=300 mm*** and height ***h=80 mm***, the web thickness is ***t=12 mm***. Determine the maximum tensile and compressive stresses in the beam due to uniform load.





$$\sum M_A = 0$$

$B_y\times3-14.4\times2.25=0$

$B_y=10.8$ KN

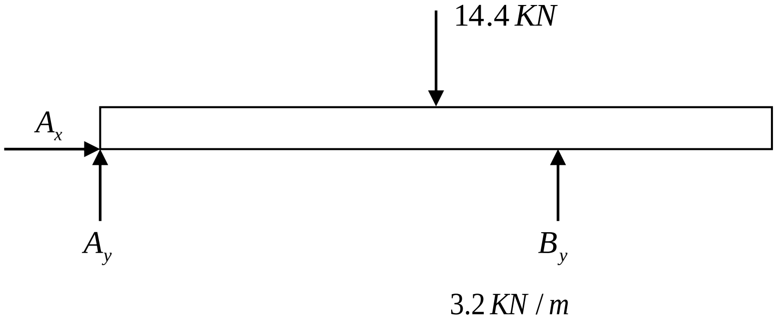
$$\sum F_y = 0$$

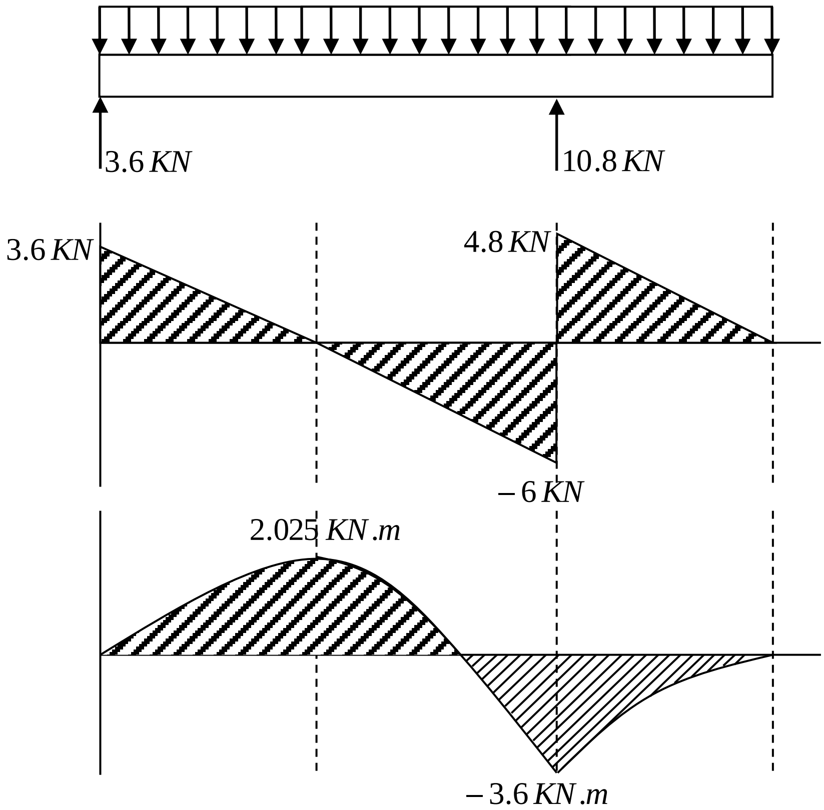
$A_y+10.8-14.4=0$

$A_y=3.6$ KN

$$\sum F_x = 0$$

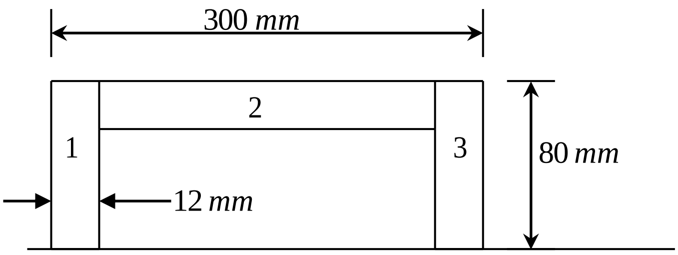
$A_x=0$





M_1=2.025 KN.m
M_2=3.6 KN.m

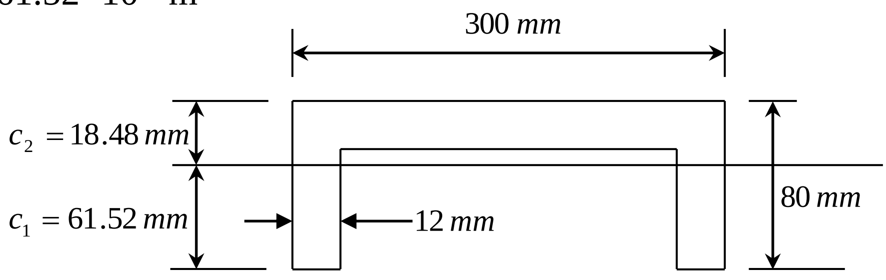
$$y_c = \frac{\sum \bar{y}A}{\sum A}$$



No. of Area	$A(m^2)$	$\bar{y}$ (m)	$\bar{y}A\,(m^3)$
1	960×10^{-6}	40×10^{-3}	38400×10^{-9}
2	3312×10^{-6}	74×10^{-3}	245088×10^{-9}
3	960×10^{-6}	40×10^{-3}	38400×10^{-9}
	$\sum A=5232\times10^{-6}$		$\sum \bar{y}A=321888\times10^{-9}$

$$y_c = \frac{321888\times10^{-9}}{5232\times10^{-6}} = 61.52\times10^{-3}\ \text{m}$$

y_c=61.52 mm



$$I_1 = \frac{bh^3}{12} + Ad^2$$

$I_1=\frac{12\times10^{-3}(80\times10^{-3})^3}{12}+960\times10^{-6}\times(21.52\times10^{-3})^2=0.95658\times10^{-6}$ m^4

$I_3= I_1=0.95658\times10^{-6}$ m^4

$I_2=\frac{bh^3}{12}+Ad^2$

$I_2=\frac{276\times10^{-3}(12\times10^{-3})^3}{12}+3312\times10^{-6}\times(12.48\times10^{-3})^2=0.55558\times10^{-6}$ m^4

$I= I_1+ I_2+ I_3=2.46874\times10^{-6}$ m^4

$$(\sigma_t)_1=\frac{M_1c_2}{I}=\frac{2.025\times10^3\times61.52\times10^{-3}}{2.46874\times10^{-6}}=50.462179 \text{ MPa}$$

$$(\sigma_t)_2=\frac{M_2c_1}{I}=\frac{3.6\times10^3\times18.48\times10^{-3}}{2.46874\times10^{-6}}=26.94815 \text{ MPa}$$

$(\sigma_t)_{max}=50.462179$ MPa

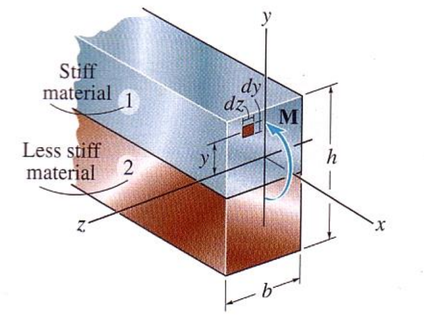
$$(\sigma_c)_1=-\frac{M_1c_1}{I}=-\frac{2.025\times10^3\times18.48\times10^{-3}}{2.46874\times10^{-6}}=-15.158339 \text{ MPa}$$

$$(\sigma_c)_2=-\frac{M_2c_2}{I}=-\frac{3.6\times10^3\times61.52\times10^{-3}}{2.46874\times10^{-6}}=-89.71054 \text{ MPa}$$

$(\sigma_c)_{max}=-89.71054$ MPa

Composite Beams:

Composite beams are made from different materials in order to efficiently carry a load.



Normal stress in material 1 is determined from $\sigma=E_1\varepsilon$

Normal stress in material 2 is determined from $\sigma=E_2\varepsilon$

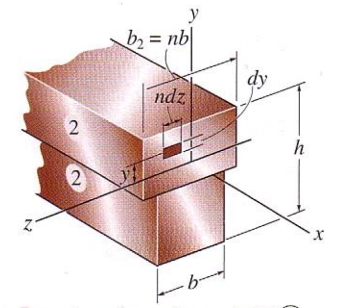
$dA=dydz$

The force dF acting on the area dA of the beam is

$dF=\sigma dA=(E_1\varepsilon)dydz$

If the material 1 is being transformed into material 2

$b_2=nb$



Beam transformed to material ②

$$d\overline{F} = \overline{\sigma}d\overline{A} = (E_2\varepsilon)ndydz$$

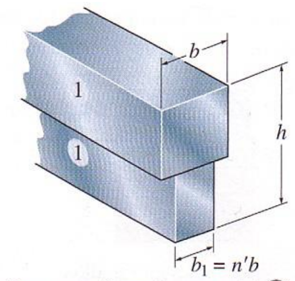
$$dF = d\overline{F}$$

$$(E_1\varepsilon)dydz = (E_2\varepsilon)ndydz$$

$$n = \frac{E_1}{E_2}$$

n: transformation factor (modular ratio).

If the material 2 is being transformed into material 1



Beam transformed to material ①

$b_1 = \overline{n}\,b$ where $\overline{n} = \frac{E_2}{E_1}$

For the transformed material

$\sigma = n\overline{\sigma}$

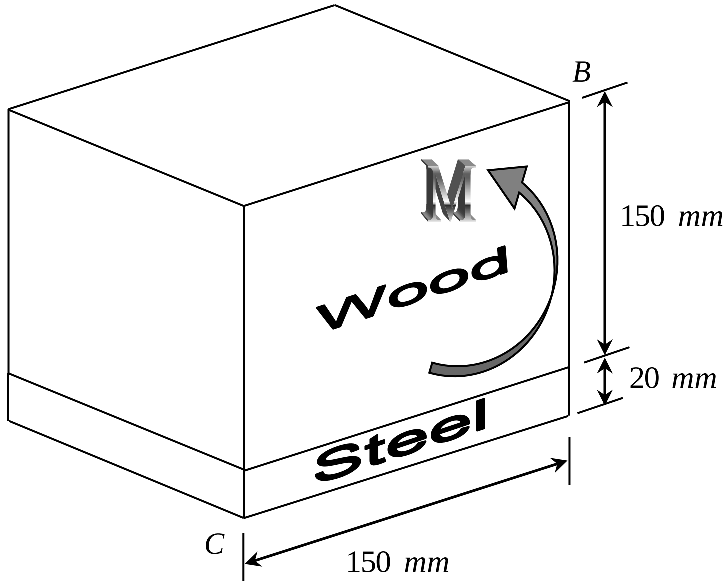
Example 46: A composite beam is made of wood and reinforced with a steel strap located on its bottom side. It has the cross sectional area shown below. If the beam is subjected to a bending moment of ***M=2 KN.m*** determine the normal stress at point ***B*** and ***C***. Take $\boldsymbol{E_w=12}$ ***GPa*** and $\boldsymbol{E_{st}=200}$ ***GPa***.

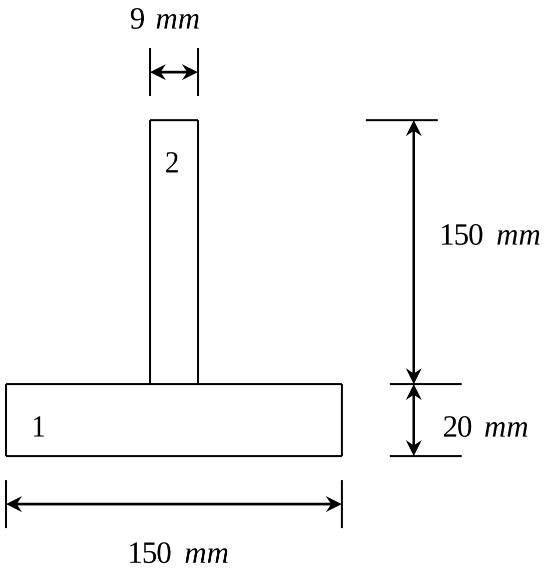
$$n=\frac{E_w}{E_{st}}$$

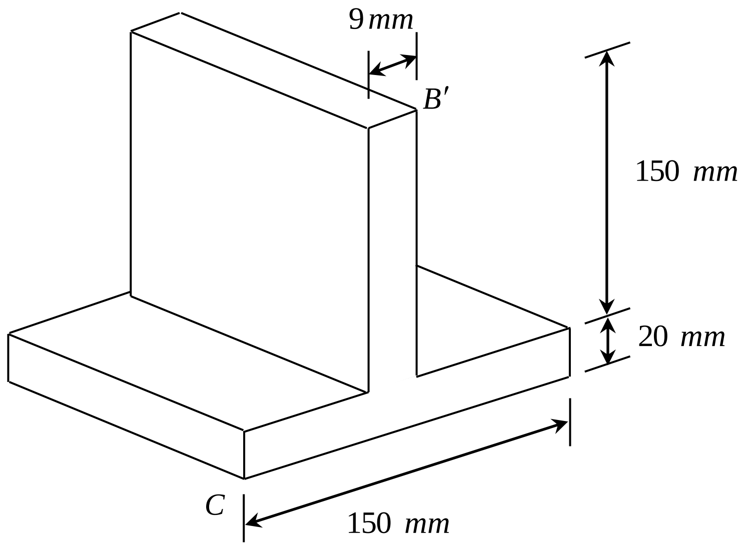
$$n=\frac{12}{200}=0.06$$

$$b_{st}=n\times b_w$$

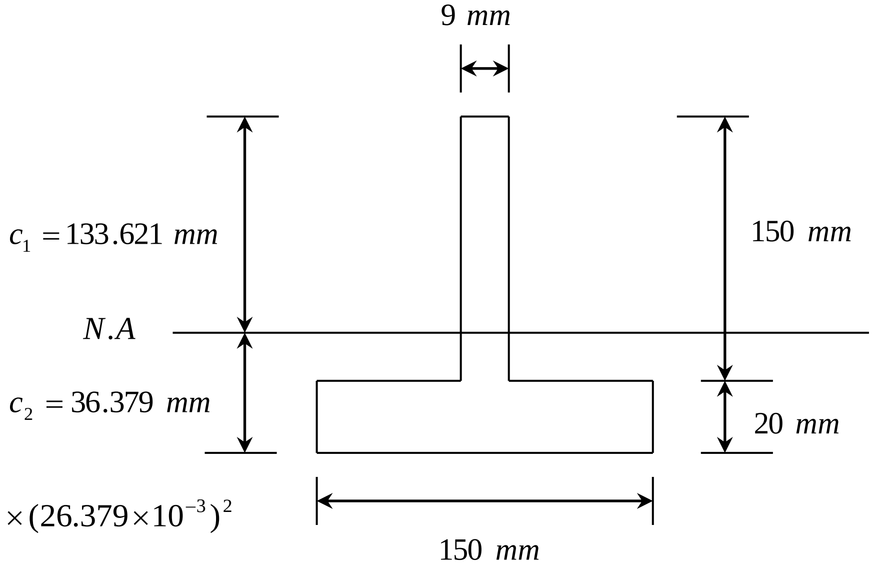
$$b_{st}=0.06\times 150=9\ mm$$







No. of Area	$A(m^2)$	$\bar{y}$ (m)	$\bar{y}A(m^3)$
1	3000×10^{-6}	10×10^{-3}	30000×10^{-9}
2	1350×10^{-6}	95×10^{-3}	128250×10^{-9}
	$\sum A=4350\times10^{-6}$		$\sum \bar{y}A=158250\times10^{-9}$



$$y_c = \frac{\sum y'A}{\sum A}$$

$$y_c = \frac{158250\times10^{-9}}{4350\times10^{-3}} = 36.379\times10^{-3}\ m$$

$$= 36.379\ mm$$

$I=I_1+I_2$

$$I_1 = \frac{bh^3}{12} + Ad^2$$

$$I_1 = \frac{150\times10^{-3}\times(20\times10^{-3})^3}{12} + 3000\times10^{-6}\times(26.379\times10^{-3})^2$$

$I_1=2.187554\times10^{-6}\ m^4$

$$I_2 = \frac{bh^3}{12} + Ad^2$$

$$I_2 = \frac{9\times10^{-3}\times(150\times10^{-3})^3}{12} + 1350\times10^{-6}\times(58.621\times10^{-3})^2$$

$I_2=7.170419\times10^{-6}\ m^4$

$I=9.35797\times10^{-6}\ m^4$

$$\sigma_{B'} = -\frac{My}{I}$$

$$\sigma_{B'} = -\frac{2\times10^3\times133.621\times10^{-3}}{9.35797\times10^{-6}} = -28.557689\ MPa$$

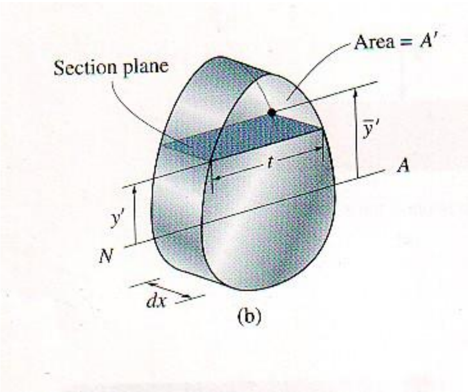
$$\sigma_B = n\times\sigma_{B'}$$

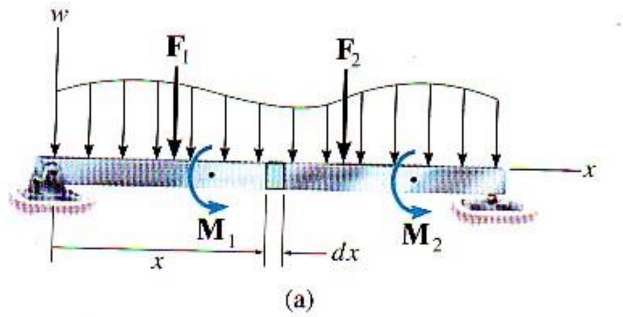
$$\sigma_B = 0.06\times(-28.557689) = -1.71346134\ MPa$$

$$\sigma_B = \frac{My}{I}$$

$$\sigma_B = \frac{2\times10^3\times36.379\times10^{-3}}{9.35797\times10^{-6}} = 7.774976\ MPa$$

Shear Stresses in Beams





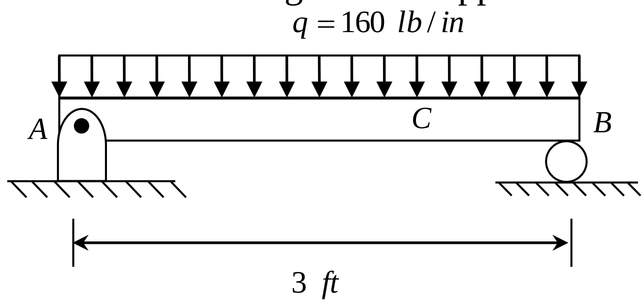
$$\tau = \frac{V}{It}\int_{A'} ydA$$

$$\int_{A'} ydA = \overline{y}'A' = Q$$

$$\tau = \frac{VQ}{It}$$

τ :- the shear stress in the member at the point located a distance y' from the neutral axis.
V:-the internal resultant shear force.
I:-the moment of inertia of the entire cross sectional area computed about the neutral axis.
t:-the width of the members cross sectional area, measured at the point where τ is to be determined.
$Q = \overline{y}'A'$, where A' is the top (or bottom) portion of the members cross sectional area, defined from the section where t is measured, and $\overline{y}'$ is the distance to the centroid of A', measured from the neutral axis.

Example 47: A metal beam with span ***L=3 ft*** is simply supported at points ***A*** and ***B***. The uniform load on the beam is ***q=160 lb/***in. The cross section of the beam is rectangular with width ***b=1 in*** and height ***h=4 in***. Determine the normal stress and shear stress at point ***C***, which is located ***1 in*** below the top of the beam and ***8 in*** from the right hand support.



$\sum M_A = 0$

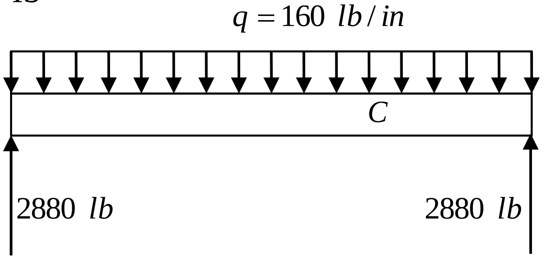
$B_y \times 3 \times 12 - 5760 \times 1.5 \times 12 = 0$

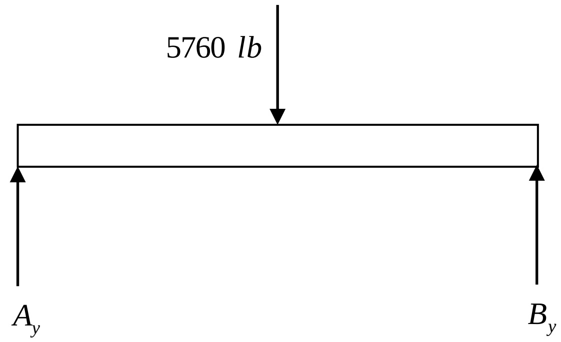
$B_y = 2880$ lb

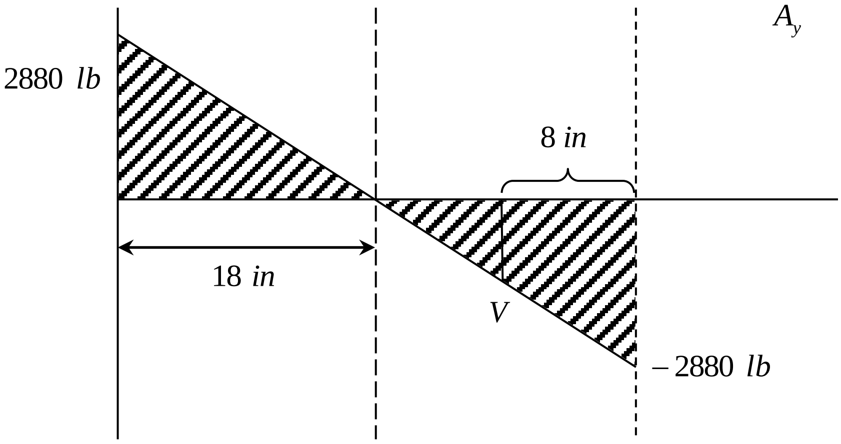
$\sum F_y = 0$

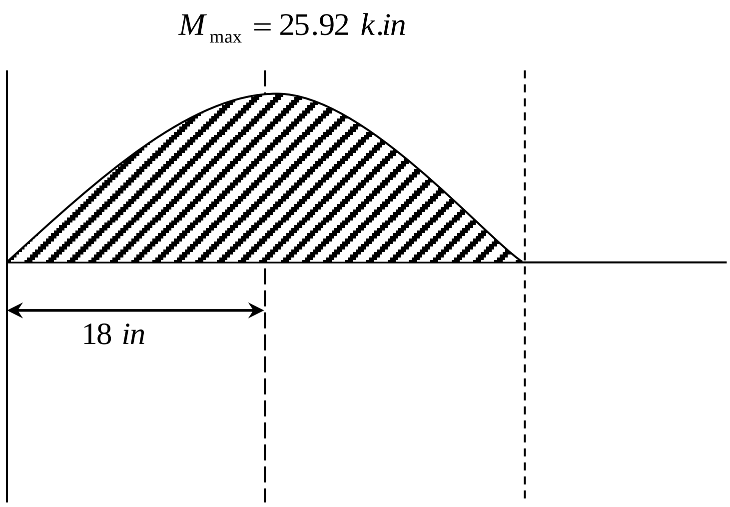
$A_y + 2880 - 5760 = 0$

$A_y = 2880$ lb





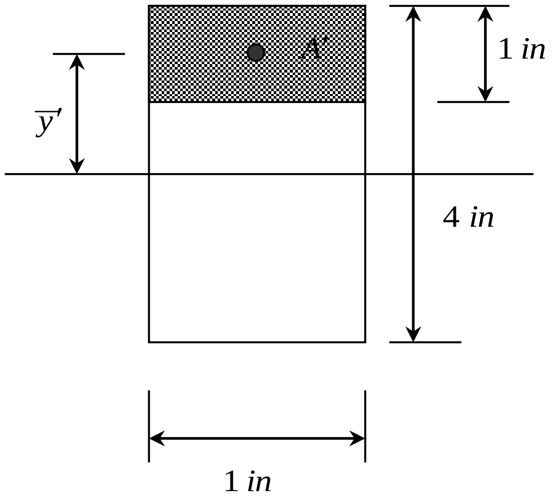




$$\frac{V}{10} = \frac{2880}{18}$$

V=1600 lb

At point C x=28 in from left end
from shear force diagram



$$M = \frac{1}{2}\times 2880\times 18 - \frac{1}{2}\times 1600\times 10$$
M=17.92 k.in

$$I = \frac{bh^3}{12} = \frac{1\times(4)^3}{12} = 5.3333\ in^4$$

A' =1×1=1 in^2

$\overline{y}'$ =1.5 in

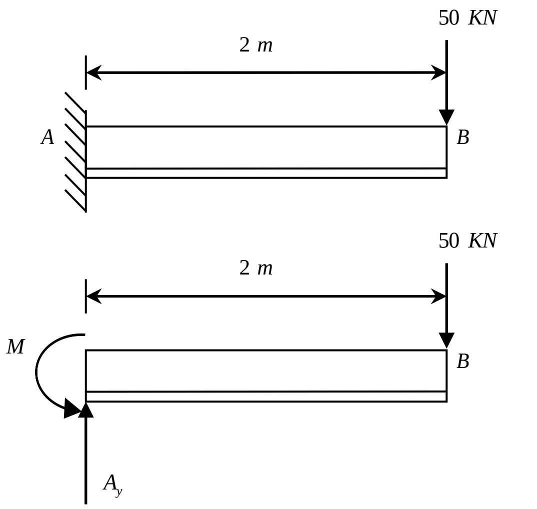
Q $= \overline{y}'A'$ =1.5×1=1.5 in^3

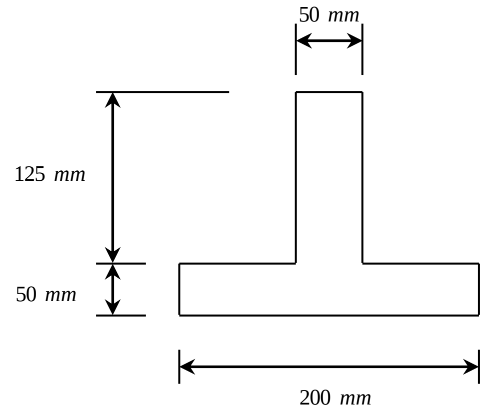
$$\sigma_C = -\frac{My}{I} = -\frac{17920\times 1}{5.3333} = -3.36\ ksi$$

$$\tau = \frac{VQ}{It}$$

$$\tau = \frac{1600\times 1.5}{5.3333\times 1} = 450\ \ psi$$

Example 48: Consider the cantilever beam subjected to the concentrated load shown below. The cross section of the beam is of T-shape. Determine the maximum shearing stress in the beam and also determine the shearing stress ***25 mm*** from the top surface of the beam of a section adjacent to the supporting wall.





$\sum M_A = 0$

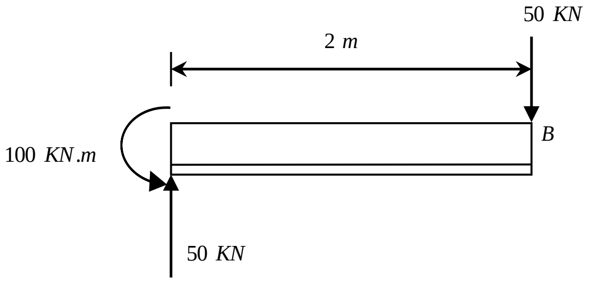
M-50×2=0

M=100 KN.m

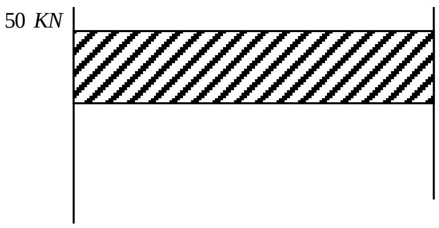
$\sum F_y = 0$

A_y-50=0

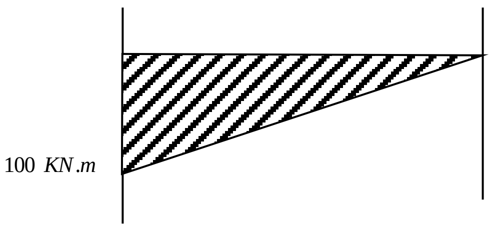
A_y=50 KN



S.F. Diagram



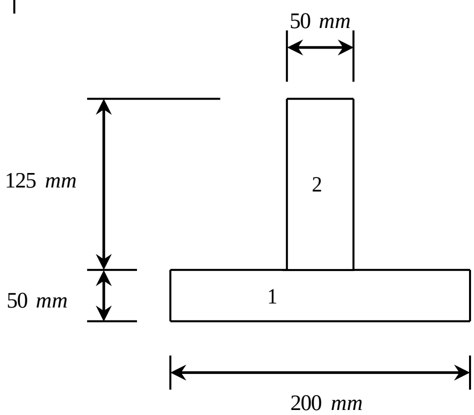
B.M. Diagram



From shear and bending moment diagrams

V=50 KN

M=100 KN.m

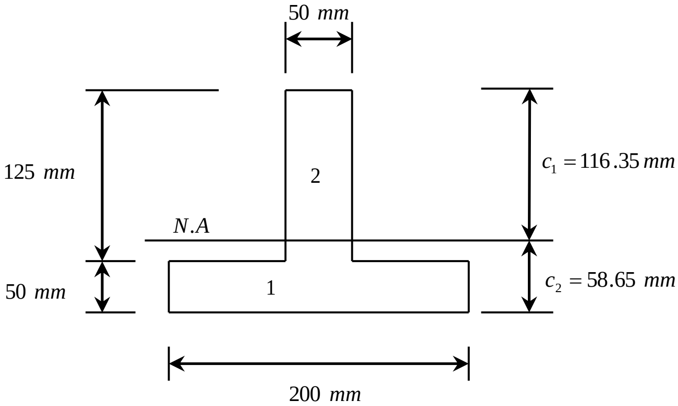


No. of Area	A(m^2)	$\bar{y}$ (m)	$\bar{y}A$ (m^3)
1	10000×10^{-6}	25×10^{-3}	250000×10^{-9}
2	6250×10^{-6}	112.5×10^{-3}	703125×10^{-9}
	$\sum A=16250\times10^{-6}$		$\sum \bar{y}A=953125\times10^{-9}$

$$y_c = \frac{\sum y'A}{\sum A}$$

$$y_c = \frac{953125\times10^{-9}}{16250\times10^{-3}} = 58.65\times10^{-3} \ \ m$$

$$= 58.65 \ mm$$



$I=I_1+I_2$

$$I_1 = \frac{bh^3}{12} + Ad^2$$

$$I_1 = \frac{200\times10^{-3}\times(50\times10^{-3})^3}{12} + 10000\times10^{-6}\times(33.65\times10^{-3})^2$$

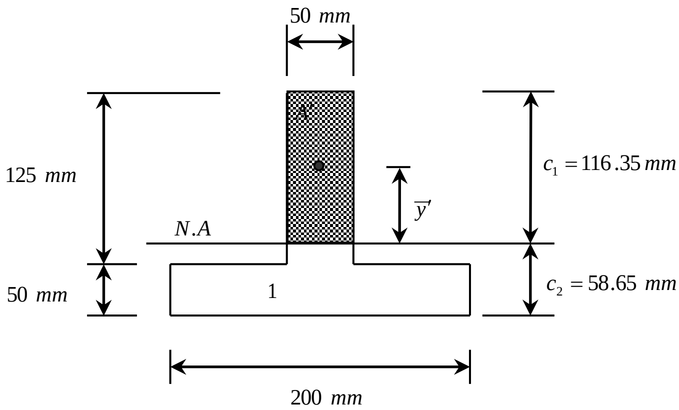
$I_1=13.40655833\times10^{-6}$ m^4

$$I_2 = \frac{bh^3}{12} + Ad^2$$

$$I_2 = \frac{50\times10^{-3}\times(125\times10^{-3})^3}{12} + 6250\times10^{-6}\times(53.85\times10^{-3})^2$$

$I_2=26.26191146\times10^{-6}$ m^4

$I=39.6684\times10^{-6}$ m^4



$Q = \bar{y}'A'$

$A' = 50\times10^{-3}\times116.35\times10^{-3}$

$= 0.0058175$ m^2

$\bar{y}' = 58.175\times10^{-3}$ m

$Q = 0.000338433$ m^3

$$\tau_{\max} = \frac{VQ}{It}$$

$$\tau_{\max} = \frac{50\times10^3\times0.000338433}{39.6684\times10^{-6}\times50\times10^{-3}} = 8.5315553\ MPa$$

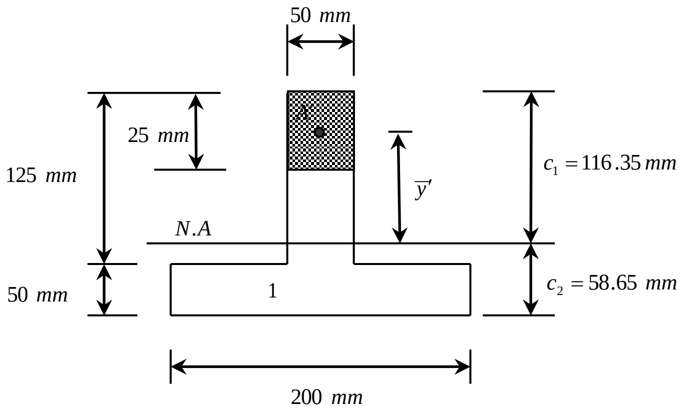
$Q = \overline{y}'A'$

$A' = 50\times10^{-3}\times25\times10^{-3}$

$= 0.00125\ m^2$

$\overline{y}' = 103.85\times10^{-3}\ m$

$Q = 0.000129812\ m^3$

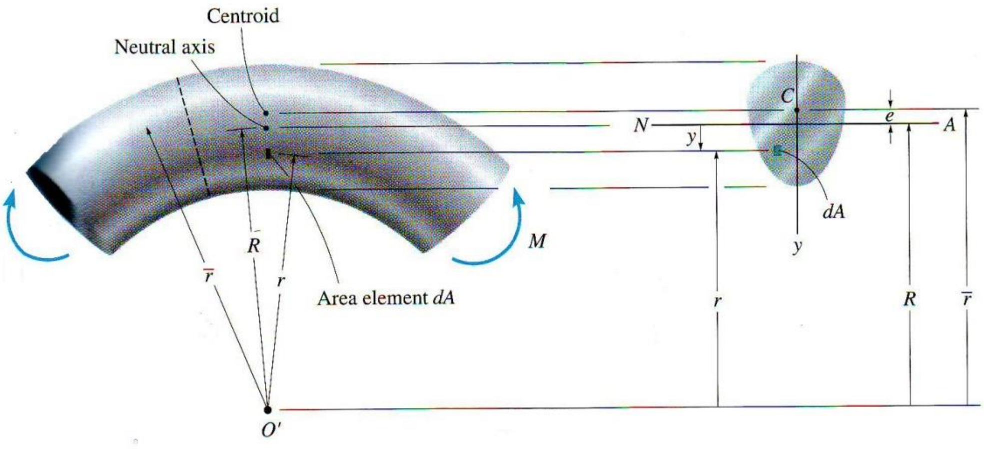


$$\tau = \frac{VQ}{It}$$

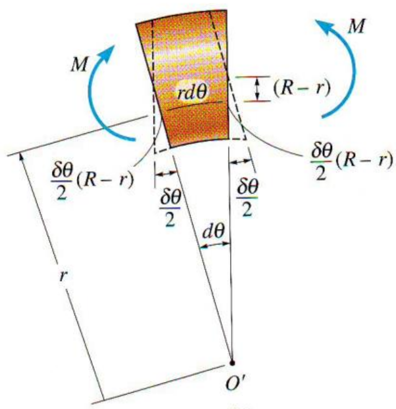
$$\tau = \frac{50\times10^{3}\times0.000129812}{39.6684\times10^{-6}\times50\times10^{-3}} = 3.272441\,MPa$$

Curved Beams

Due to the curvature of the beam, the normal strain in the beam does not vary linearly with depth as in the case of a straight beam .As result, the neutral axis does not pass through the centroid of the cross section.



If we isolate a differential segment of the beam let a strip material located at r distance has an original length r dθ .Due to the rotations δθ/2, the strip's total change in length is δθ)R-r(



(b)

$$\varepsilon = \frac{\delta\theta(R-r)}{rd\theta} \quad ,k = \frac{\delta\theta}{d\theta} \quad \Rightarrow \; \varepsilon = k(\frac{R-r}{r})$$

Strain is a nonlinear function of r, in fact it varies in a hyperbolic fashion .Hooke's law applies,

$$\sigma = Ek(\frac{R-r}{r})$$

$$\sum F_R = 0$$

$$\int_A \sigma dA = 0 \; \Rightarrow \int_A Ek(\frac{R-r}{r})dA = 0$$

$$R\int_A \frac{dA}{r} - \int_A dA = 0$$

$$R = \frac{A}{\int_A \frac{dA}{r}}$$

R-:The location of the neutral axis, specified from the center of curvature $0'$ of the member.

A-:The cross -sectional area of the member.

r -:The arbitrary position of the area element dAon the cross section, specified from the center of curvature $0'$ of the member.

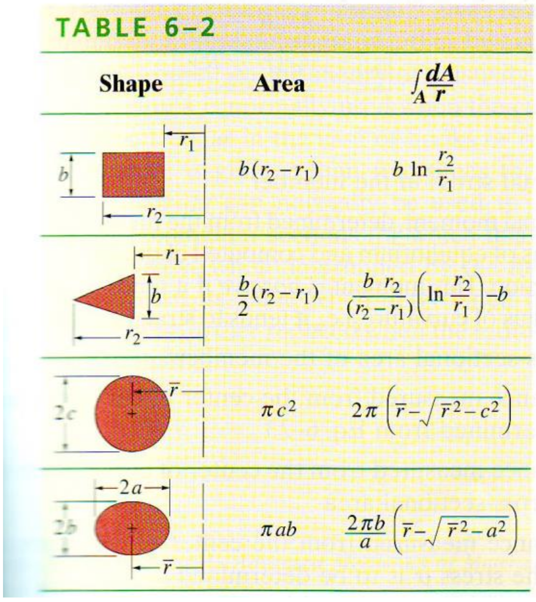


TABLE 6–2

Shape	Area	$\int_A \frac{dA}{r}$
Rectangle (b, r_1, r_2)	$b(r_2 - r_1)$	$b \ln \frac{r_2}{r_1}$
Triangle (b, r_1, r_2)	$\frac{b}{2}(r_2 - r_1)$	$\frac{b\, r_2}{(r_2 - r_1)}\left(\ln \frac{r_2}{r_1}\right) - b$
Circle ($2c$, $\bar{r}$)	πc^2	$2\pi\left(\bar{r} - \sqrt{\bar{r}^2 - c^2}\right)$
Ellipse ($2a$, $2b$, $\bar{r}$)	πab	$\frac{2\pi b}{a}\left(\bar{r} - \sqrt{\bar{r}^2 - a^2}\right)$

$$\sigma = \frac{My}{Ae(R - y)}$$

y=R-r , e-$\bar{r}$ =R

σ -:The normal stress in the member.

M -:The internal moment, determined from the method of sections equations of equilibrium and computed about the centroidal axis .

A -:The cross-sectional area of the member.

R -:The distance measured from the center of curvature to the neutral axis.

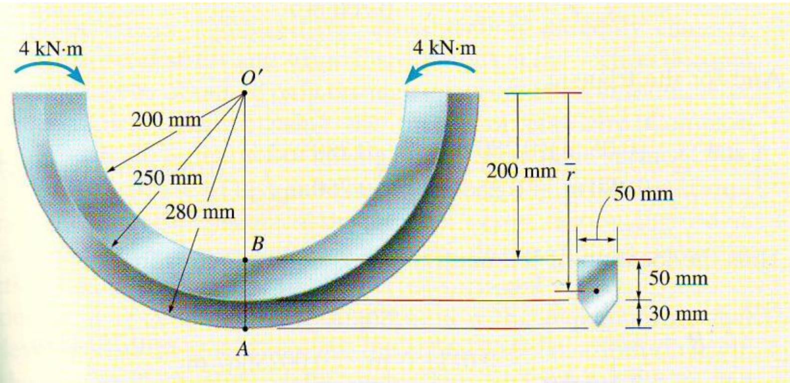
$\bar{r}$ -:The distance measured from the center of curvature to the centroid of the cross-sectional area.

r -:The distance measured from the center of curvature to the point where the stress σ is to be determined.

$$\sigma_o = \frac{M(r_o - R)}{Ar_o(\bar{r} - R)}$$ Normal stress at the bar's top.

$$\sigma_i = \frac{M(R - r_i)}{Ar_i(\bar{r} - R)}$$ Normal stress at the bar's bottom.

Example:-The curved bar has a cross-sectional area shown below .If it is subjected to bending moments of 4 kN • m, determine the maximum normal stress developed in the bar.



Area	A(mm^2)	$\bar{y}$(mm)	$\bar{y}$A(mm^3)
rectangle	2500	225	562500
triangle	750	260	195000
	3250		757500

$$\bar{r} = \frac{\sum \bar{y}A}{\sum A} = \frac{757500}{3250} = 233.076\ mm$$

$$R = \frac{A}{\int_A \frac{dA}{r}}$$

$$\sum A = 3250\ \ mm$$

$$\sum \int \frac{dA}{R} = b\ln\frac{r_2}{r_1} + \frac{br_2}{(r_2 - r_1)}\ln\frac{r_2}{r_1} - b$$

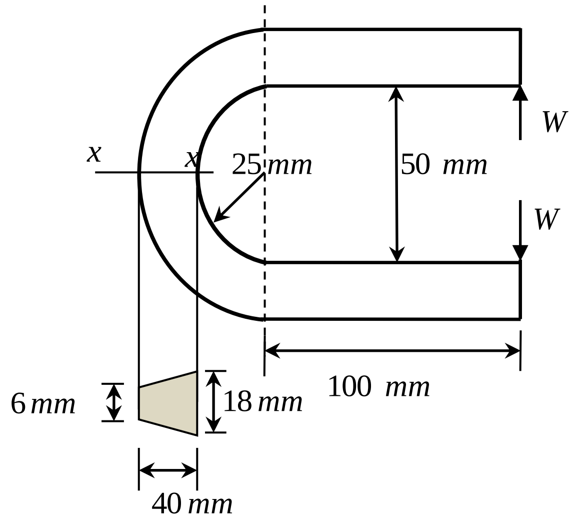
$$= 50\ln\frac{250}{200} + \frac{50 \times 280}{(280 - 250)}\ln\frac{280}{250} - 50 = 14.04389\ mm$$

$$R = \frac{3250}{14.04389} = 231.417\ mm$$

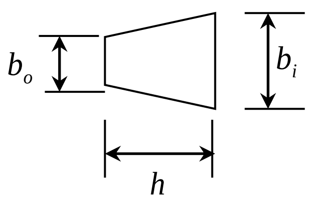
$$\sigma_B = -\frac{M(R-r_i)}{Ar_i(\bar{r}-R)} = -\frac{4\times10^3(231.417\times10^{-3}-200\times10^{-3})}{3250\times10^{-6}\times200\times10^{-3}(233.076\times10^{-3}-231.417\times10^{-3})} = -116.5373\,MPa$$

$$\sigma_A = \frac{M(r_o-R)}{Ar_o(\bar{r}-R)} = \frac{4\times10^3(280\times10^{-3}-231.417\times10^{-3})}{3250\times10^{-6}\times280\times10^{-3}(233.076\times10^{-3}-231.417\times10^{-3})} = 128.7231\,MPa$$ The maximum stress at point A=$128.7231\ MPa$

Example:-The frame of a punch press is shown below. Find the stresses at the inner and outer surface at section x-x of the frame if W=5000 N.



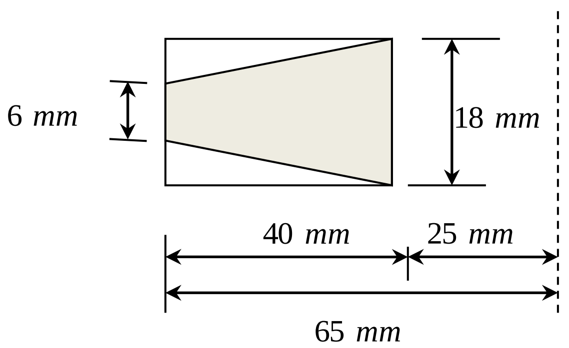
$$A = \frac{b_i + b_o}{2}h$$



$$A = (\frac{18+6}{2})\times40 = 480\ mm^2$$

$$\int\frac{dA}{R} = (\frac{b_i r_o - b_o r_i}{h})\ln\frac{r_o}{r_i} - (b_i - b_o) = (\frac{18\times65-6\times25}{40})\ln\frac{65}{25} - (18-6) = 12.365\ mm$$

$$R = \frac{A}{\int_A \frac{dA}{r}} = \frac{480}{12.365} = 38.8175\ mm$$



Area(mm^2)	A(mm^2)	$\bar{y}$(mm)	$\bar{y}A$(mm^3)
rectangle	720	45	32400
Triangle	-120	51.666	-6200
Triangle	-120	51.666	-6200
	480		20000

$$\bar{r} = \frac{\sum \bar{y}A}{\sum A} = \frac{20000}{480} = 41.666\ mm$$

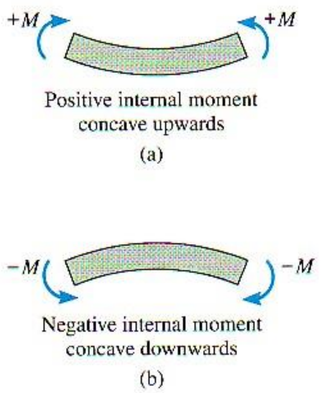
M=W×d=5000×(100×10^{-3}+41.666×10^{-3})=708.33 N.m

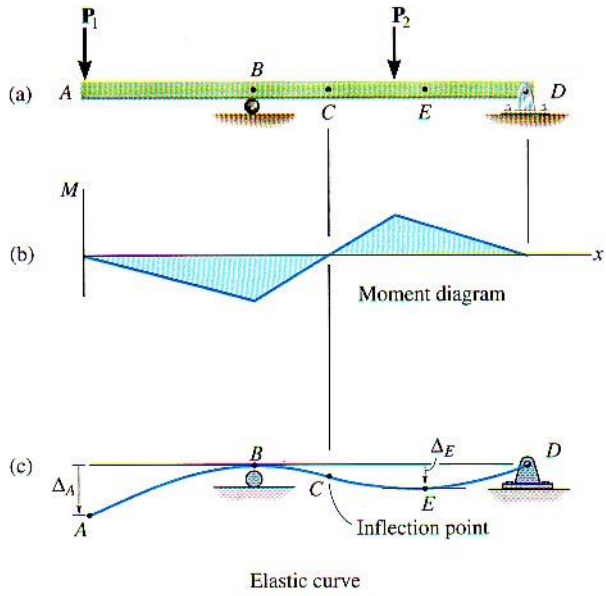
$$\sigma_i = \frac{M(R-r_i)}{Ar_i(\bar{r}-R)} + \frac{W}{A} = \frac{708.33(38.8175\times10^{-3} - 25\times10^{-3})}{480\times10^{-6}\times25\times10^{-3}(41.666\times10^{-3} - 38.8175\times10^{-3})} + \frac{5000}{480\times10^{-6}}$$
$$= 296.747\ MPa$$

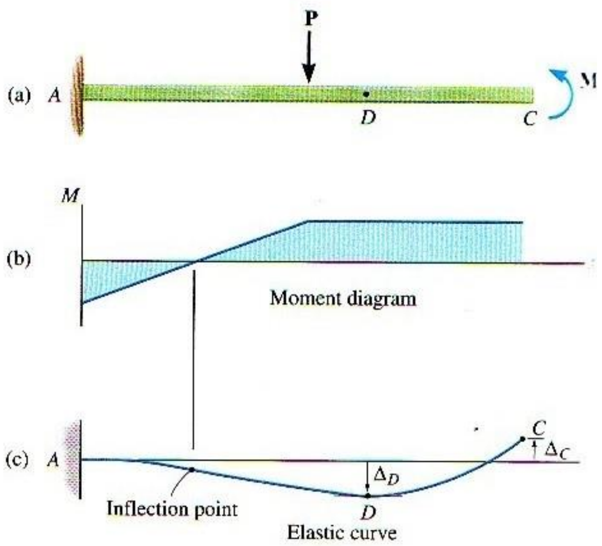
$$\sigma_o = -\frac{M(r_o-R)}{Ar_o(\bar{r}-R)} + \frac{W}{A} = -\frac{708.33(65\times10^{-3} - 38.8175\times10^{-3})}{480\times10^{-6}\times65\times10^{-3}(41.666\times10^{-3} - 38.8175\times10^{-3})} + \frac{5000}{480\times10^{-6}}$$
$$= -198.26\,MPa$$

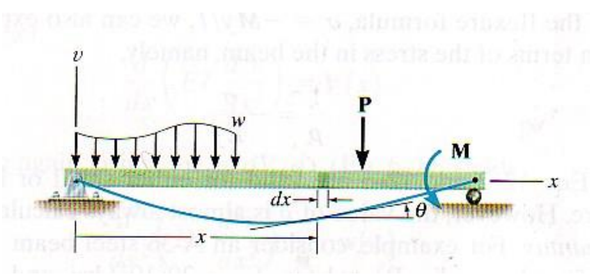
Slop and Deflection in Beams

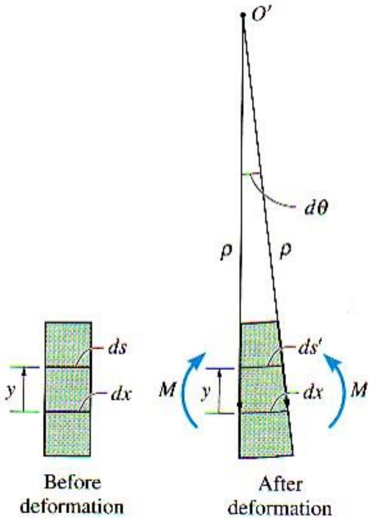
The elastic curve :-the deflection diagram of the longitudinal axis that passes through the centroid of each cross sectional area of the beam.











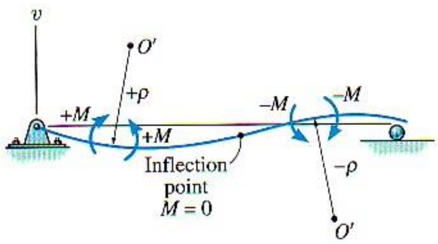
x-axis extends positive to the right.
v-axis extends positive upward from the x-axis.

$$\frac{1}{\rho}=-\frac{\varepsilon}{y}$$

$$\varepsilon=\frac{\sigma}{E}$$

$$\sigma=-\frac{My}{I}$$

$$\frac{1}{\rho}=\frac{M}{EI}$$



When M is positive, ρ extends above the beam, i.e. ρ in the positive v direction. When M is negative, ρ extends below the beam, or in the negative v direction.

Integration Method

The elastic curve for a beam can be expressed mathematically as v=f(x)

$$\frac{1}{\rho}=\frac{d^2v/dx^2}{[1+(dv/dx)^2]^{3/2}}$$

$$\frac{M}{EI}=\frac{d^2v/dx^2}{[1+(dv/dx)^2]^{3/2}}$$

The slop of the elastic curve which is determined from dv/dx will be very small, and its square will be negligible compared with unity.

$$\frac{1}{\rho}=\frac{d^2v}{dx^2}$$

$$\frac{M}{EI}=\frac{d^2v}{dx^2}$$

$$V = \frac{dM}{dx}$$

$$V(x) = \frac{d}{dx}(EI\frac{d^2v}{dx^2})$$

$$-w = \frac{dV}{dx}$$

$$-w(x) = \frac{d^2}{dx^2}(EI\frac{d^2v}{dx^2})$$

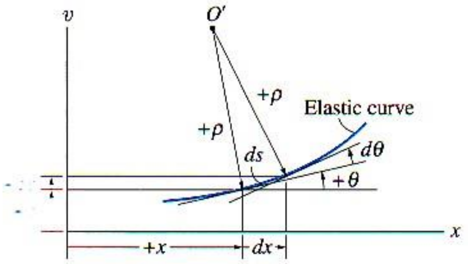
EI always positive quantity

$$M = EI(\frac{d^2v}{dx^2})$$

$$V = EI(\frac{d^3v}{dx^3})$$

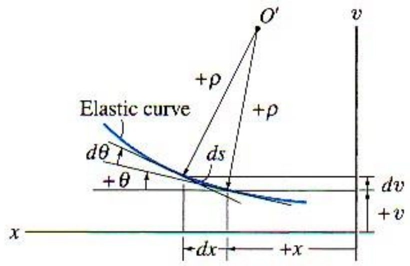
$$-w = EI(\frac{d^4v}{dx^4})$$

Sign convention and coordinates



Positive sign convention

(b)



Positive sign convention

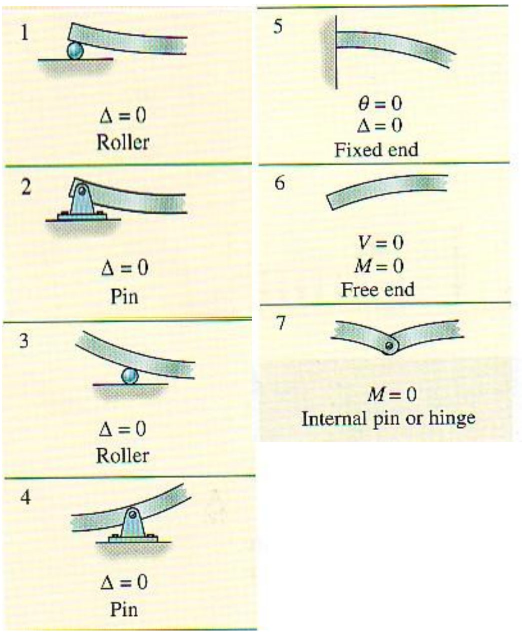
(c)

Positive deflection v is upward, the positive slope θ will be measured counterclockwise from the x-axis when x is positive to the right.

If positive x is directed to the left, then θ will be positive clockwise.

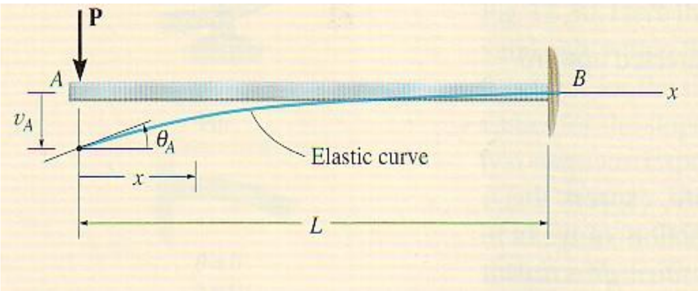
$$\theta = \frac{dv}{dx}$$

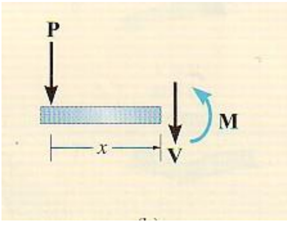
Boundary conditions



$$EI\frac{dv}{dx} = \int Mdx + C_1$$
$$EIv = \int[\int Mdx]dx + C_1x + C_2$$

Example: The cantilevered beam shown is subjected to a vertical load P at its end. Determine the equation of elastic curve. EI is constant.





$\sum M = 0$

M+Px=0

M=-Px

$M = EI(\frac{d^2v}{dx^2})$=-Px

$EI\frac{dv}{dx} = -\frac{1}{2}Px^2 + C_1$ ……………………..(1)

$EIv = -\frac{1}{6}Px^3 + C_1x + C_2$ ………………………(2)

Boundary conditions

at x=L

$$\theta = \frac{dv}{dx} = 0 \quad \text{and} \quad v = 0$$

$$0 = -\frac{1}{2}PL^2 + C_1 \Longrightarrow C_1 = \frac{1}{2}PL^2$$

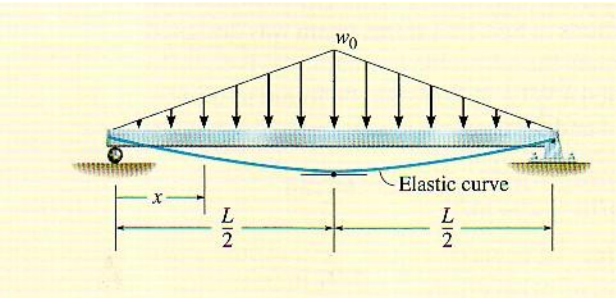
$$0 = -\frac{1}{6}PL^3 + \frac{1}{2}PL^3 + C_2 \Longrightarrow C_2 = \frac{1}{6}PL^3 - \frac{1}{2}PL^3 = -\frac{PL^3}{3}$$

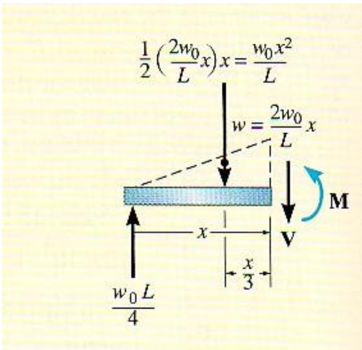
$$EI\theta = -\frac{1}{2}Px^2 + \frac{1}{2}PL^2 \Longrightarrow \theta = \frac{P}{2EI}(L^2 - x^2)$$

$$EIv = -\frac{1}{6}Px^3 + \frac{PL^2}{2}x - \frac{PL^3}{3}$$

$$v = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3)$$

Example: The simply supported beam shown supports the triangular distributed loading. Determine the maximum deflection. EI is constant.





Due to symmetry we take $0 \le x \le \frac{L}{2}$

$$\sum M = 0$$

$$M - \frac{w_\circ L}{4}x + \frac{w_\circ x^3}{3L} = 0 \Longrightarrow M = w_\circ\left(\frac{L}{4}x - \frac{x^3}{3L}\right)$$

$$M = EI\left(\frac{d^2v}{dx^2}\right) = w_\circ\left(\frac{L}{4}x - \frac{x^3}{3L}\right)$$

$$EI\frac{dv}{dx} = \frac{w_\circ L}{8}x^2 - \frac{w_\circ x^4}{12L} + C_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$EIv = \frac{w_\circ L}{24}x^3 - \frac{w_\circ x^5}{60L} + C_1x + C_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Boundary conditions

at $\boldsymbol{x}=\frac{L}{2}$ $\qquad \theta=\frac{dv}{dx}=0$

$$0=\frac{w_{\circ}L}{8}x^{2}-\frac{w_{\circ}x^{4}}{12L}+C_{1} \qquad\qquad C_{1}=-\frac{5w_{\circ}L^{3}}{192}$$

<u>***at x=0 v=0***</u>

$$0=0-0+0+C_{2} \qquad\qquad C_{2}=0$$

$$v=\frac{w_{\circ}}{EI}\left(\frac{L}{24}x^{3}-\frac{x^{5}}{60L}-\frac{5L^{3}}{192}x\right)$$

V_{max} at $\boldsymbol{x}=\frac{L}{2}$

$$v_{\max}=\frac{w_{\circ}}{EI}\left(\frac{L^{4}}{192}-\frac{L^{4}}{1920}-\frac{5L^{4}}{384}\right)=-\frac{w_{\circ}}{120}\frac{L^{4}}{EI}$$

<u>Discontinuity Method</u>

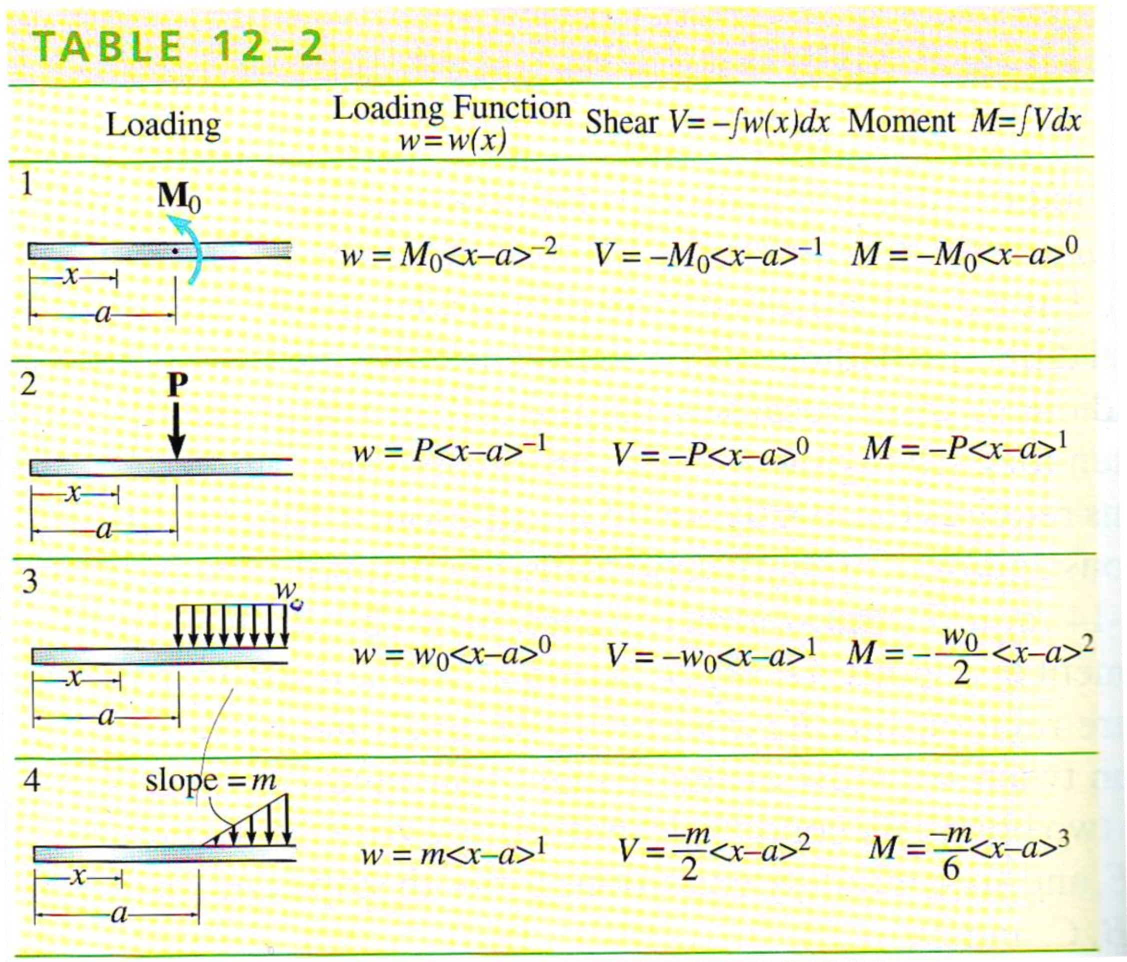


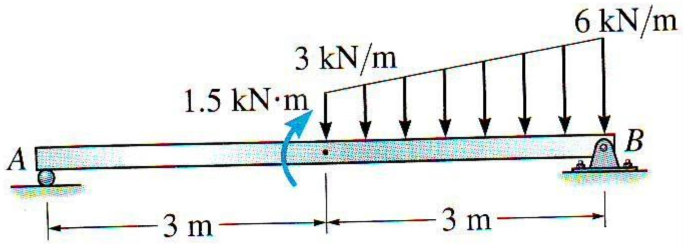
TABLE 12-2

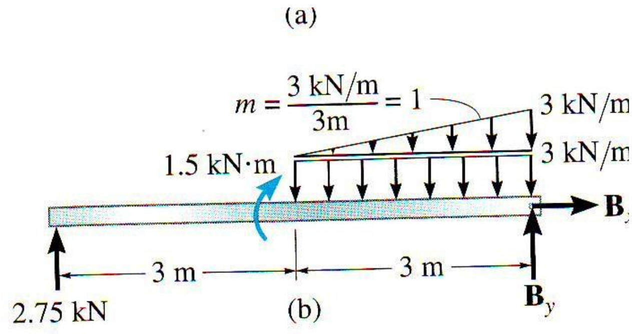
	Loading	Loading Function $w=w(x)$	Shear $V=-\int w(x)dx$	Moment $M=\int V dx$
1	$\mathbf{M}_0$, x, a	$w=M_0\langle x-a\rangle^{-2}$	$V=-M_0\langle x-a\rangle^{-1}$	$M=-M_0\langle x-a\rangle^{0}$
2	$\mathbf{P}$, x, a	$w=P\langle x-a\rangle^{-1}$	$V=-P\langle x-a\rangle^{0}$	$M=-P\langle x-a\rangle^{1}$
3	w_0, x, a	$w=w_0\langle x-a\rangle^{0}$	$V=-w_0\langle x-a\rangle^{1}$	$M=-\frac{w_0}{2}\langle x-a\rangle^{2}$
4	slope $=m$, x, a	$w=m\langle x-a\rangle^{1}$	$V=\frac{-m}{2}\langle x-a\rangle^{2}$	$M=\frac{-m}{6}\langle x-a\rangle^{3}$

$$\langle x-a\rangle^n=\begin{cases}0 & \text{for } x<a\\ (x-a)^n & \text{for } x\ge a\end{cases}$$

$$\int\langle x-a\rangle^n dx=\frac{\langle x-a\rangle^{n+1}}{n+1}+C$$

Example:- Find the moment expression using continuity equations.

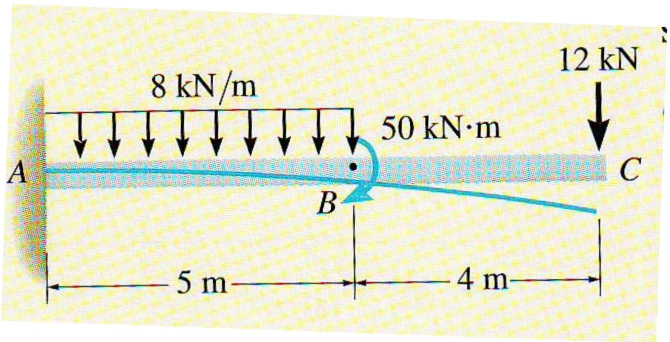


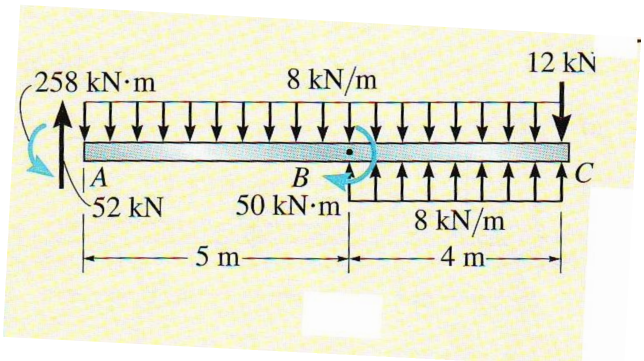


$M=2.75\langle x-0\rangle^1+1.5\langle x-3\rangle^0-\frac{3}{2}\langle x-3\rangle^2-\frac{1}{6}\langle x-3\rangle^3$

$=2.75x+1.5\langle x-3\rangle^0-\frac{3}{2}\langle x-3\rangle^2-\frac{1}{6}\langle x-3\rangle^3$

Example:- Determine the equation of the elastic curve for the beam shown below. EI is constant.





$\sum F_y=0$

$A_y-40-12=0 \quad \Rightarrow \quad A_y=52$ kN

$\sum M_A=0$

$M_A-40\times2.5-50-12\times9=0 \quad \Rightarrow \quad M_A=258$ kN.m

$M=-258<x-0>^0+52<x-0>^1-\frac{8}{2}<x-0>^2+50<x-5>^0+\frac{8}{2}<x-5>^2$

$EI\frac{d^2v}{dx^2}=-258+52x-4x^2+50<x-5>^0+4<x-5>^2$

$EI\frac{dv}{dx}=-258x+26x^2-\frac{4}{3}x^3+50<x-5>^1+\frac{4}{3}<x-5>^3+C_1$……………..…(1)

$EIv=-129x^2+\frac{26}{3}x^3-\frac{1}{3}x^4+25<x-5>^2+\frac{1}{3}<x-5>^4+C_1x+C_2$…………(2)

<u>B.C</u>

$\frac{dv}{dx}=0$ at x=0 in eq.(1)

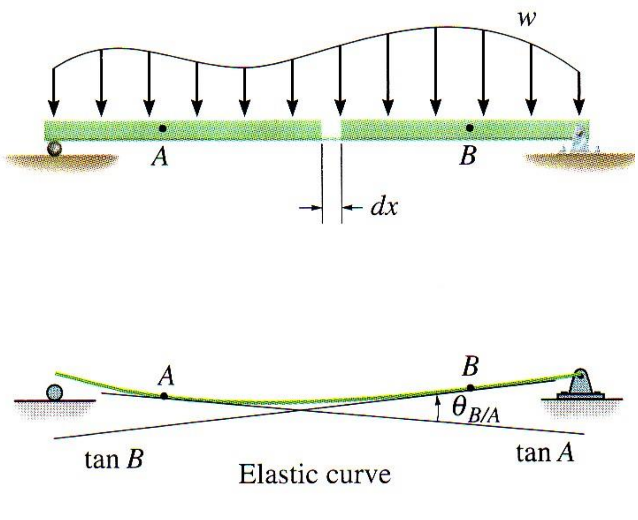
v=0 at x=0 in eq.(2)

$C_1=0$

$C_2=0$

$v=\frac{1}{EI}(-129x^2+\frac{26}{3}x^3-\frac{1}{3}x^4+25<x-5>^2+\frac{1}{3}<x-5>^4)$

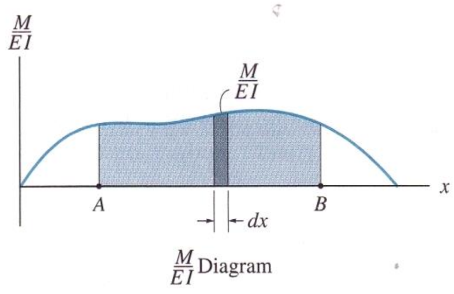
<u>Moment Area Method</u>



(a)

$$\theta_{B/A}=\int_A^B\frac{M}{EI}dx$$

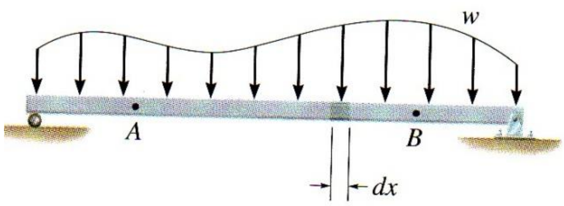
The notation $\theta_{B/A}$ is referred to as the angle of the tangent at B measured with respect to the tangent at A.

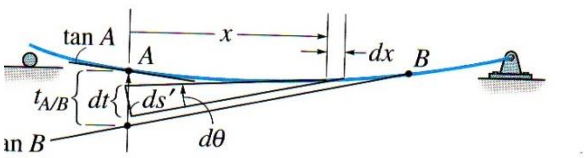


<u>Theorem 1</u> The angle between the tangents at any two points on the elastic curve equals the area under the M/EI diagram between these two points.

If the area under M/EI diagram is positive, the angle is measured counterclockwise from the tangent A to tangent B.

If the area under M/EI diagram is negative, the angle $\theta_{B/A}$ is measured clockwise from tangent A to tangent B. $\theta_{B/A}$ will measured in radians.





$$t_{A/B} = \int_A^B x \frac{M}{EI} dx$$

$t_{A/B}$: the vertical deviation of the tangent at A with respect to the tangent at B.

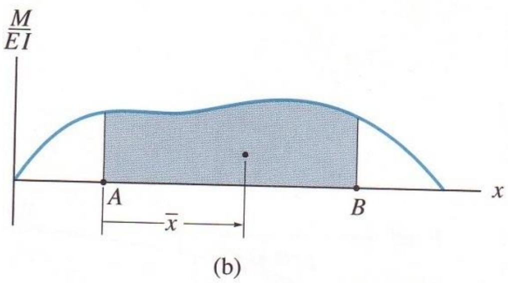
$$\int x dA = \bar{x} \int dA$$

$\int_A^B \frac{M}{EI} dx$ represents the area under the M/EI diagram, we can also write:-

$$t_{A/B} = \bar{x} \int_A^B \frac{M}{EI} dx$$

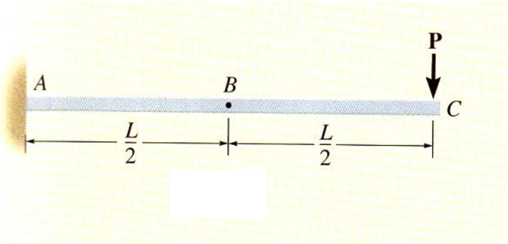
$\bar{x}$ is the distance from A to the centroid of the area under the M/EI diagram between A and B.

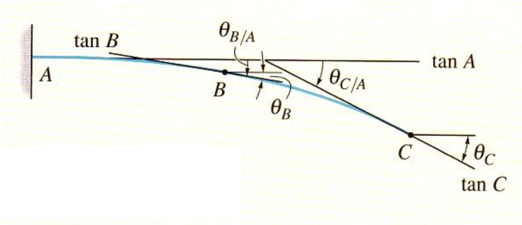
Theorem 2 The vertical deviation of the tangent at a point A on the elastic curve with respect to the tangent extended from another point B equals the moment of the area under the M/EI diagram between these two points. This moment is computed about point A where the vertical deviation $t_{A/B}$ is to be determined.



(b)

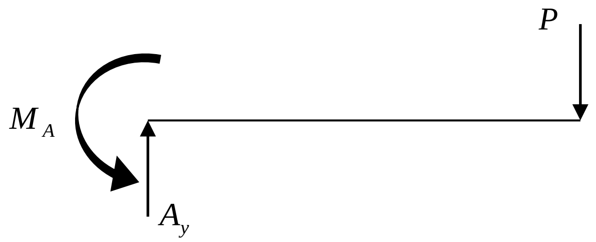
Example: Determine the slope of the beam shown at points ***B*** and ***C***. ***EI*** is constant





$\theta_B = \theta_{B/A}$

$\theta_C = \theta_{C/A}$



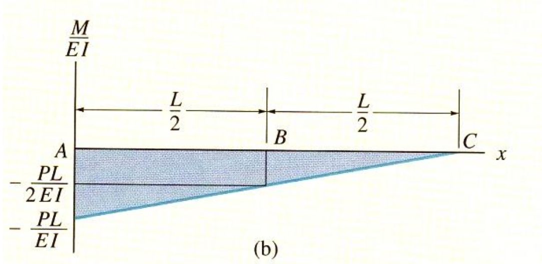
$\sum M_A = 0$

$M_A - PL = 0$

$M_A = PL$

$\sum F_y = 0$

$A_y = P$



(b)

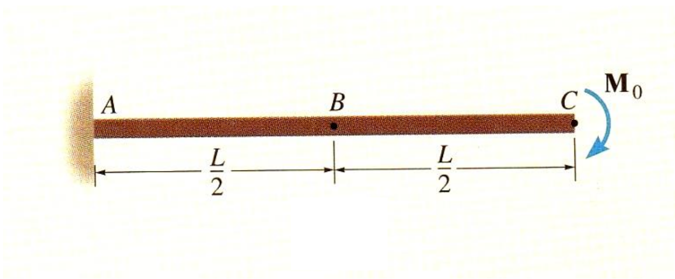
$\theta_{B/A} = \int_A^B \frac{M}{EI} dx$ Area under the M/EI diagram from A to B

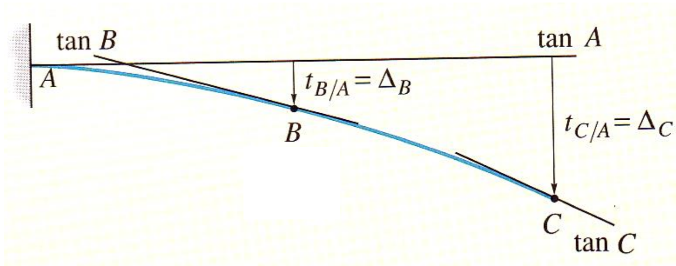
$$\theta_B = \theta_{B/A} = \left(\frac{-PL}{2EI}\right)\left(\frac{L}{2}\right) + \frac{1}{2}\left(\frac{-PL}{EI} + \frac{PL}{2EI}\right)\left(\frac{L}{2}\right)$$

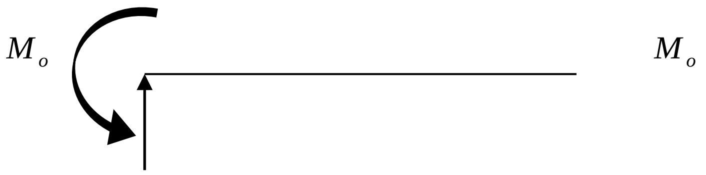
$$= \frac{-3PL^2}{8EI} \text{ rad} \quad \text{clockwise}$$

$$\theta_C = \theta_{C/A} = \left(\frac{-PL}{EI}\right)(L)\frac{1}{2} = \frac{-PL^2}{2EI} \quad \text{rad} \quad \text{clockwise}$$

Example: Determine the displacement of points B and C of the beam shown. EI is constant.





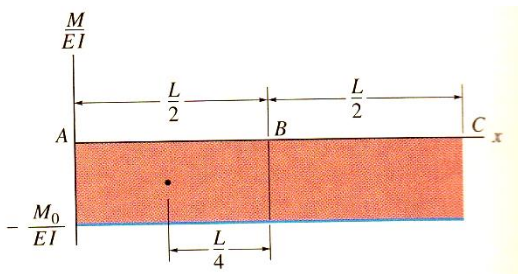


$v_B = t_{B/A}$

$v_C = t_{C/A}$

$\sum F_y = 0$

$A_y = 0$



$\sum M_A = 0$

$M - M_o = 0$

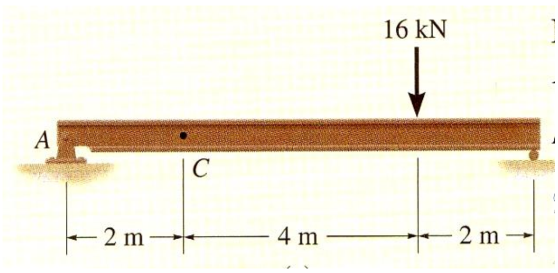
$M = M_o$

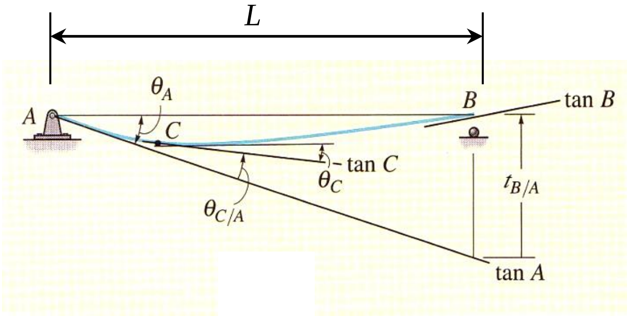
$$v_B = t_{B/A} = \bar{x}\int_A^B \frac{M}{EI}dx$$

$$= \frac{1}{2}\left(\frac{L}{2}\right)\left(\frac{-M_o}{EI}\right)\left(\frac{L}{2}\right) = \frac{-M_oL^2}{8EI}$$

$$v_C = t_{C/A} = \bar{x}\int_C^A \frac{M}{EI}dx = \left(\frac{L}{2}\right)\left(\frac{-M_o}{EI}\right)(L) = \frac{-M_oL^2}{2EI}$$

Example: Determine the slope at point ***C*** for the steel beam shown. Take $\boldsymbol{E_{st}=200\ GPa}$, $\boldsymbol{I=17\times10^6\ mm^4}$

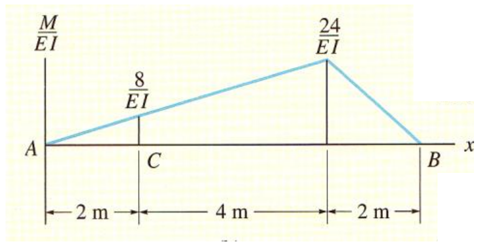




θ_C

$$\theta_C = |\theta_A| - |\theta_{C/A}|$$

Since the angle is very small $\theta_A = \tan\theta_A = \dfrac{t_{B/A}}{L}$



$$t_{B/A} = \left(2+\frac{1}{3}(6)\right)\left(\frac{24}{EI}\right)(6)\left(\frac{1}{2}\right)+\frac{2}{3}(2)\left(\frac{24}{EI}\right)(2)\left(\frac{1}{2}\right)$$

$$t_{B/A} = \frac{320}{EI}$$

$$\theta_{C/A} = \left(\frac{8}{EI}\right)(2)\left(\frac{1}{2}\right) = \frac{8}{EI}$$

$$\theta_C = \frac{320}{8EI} - \frac{8}{EI} = \frac{32}{EI}$$

$$\theta_C = \frac{32}{200\times10^9\times17\times10^{-6}} = 0.009411 \text{ rad}$$

Castigliano's Theorem Applied to Beams

$$\Delta = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI}$$

Where:-

Δ=displacement of the point caused by the real loads acting on the beam.

P=external force of variable magnitude applied to the beam in the direction of Δ.

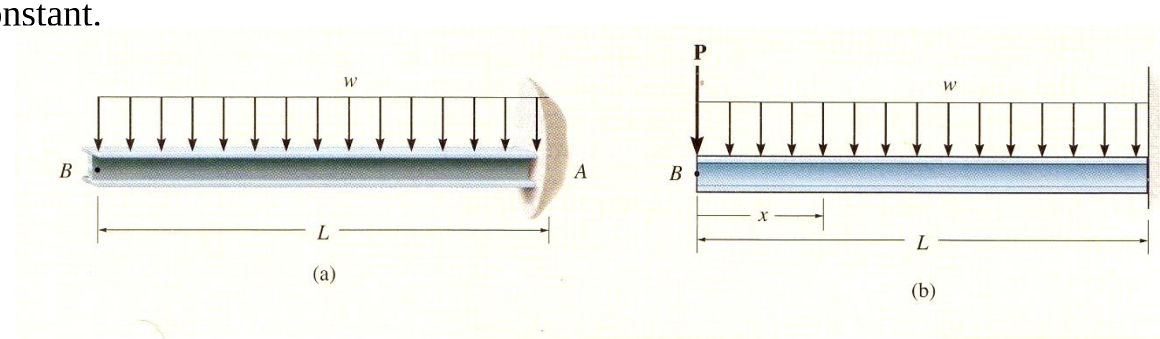
M=internal moment in the beam, expressed as a function of x and caused by both the force P and the load on the beam.

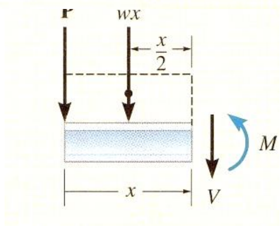
$$\theta = \int_0^L M\left(\frac{\partial M}{\partial \overline{M}}\right)\frac{dx}{EI}$$

θ=the slope of the tangent at a point on the elastic curve.

$\overline{M}$ =an external couple moment acting at the point.

Example: Determine the displacement of point B on the beam shown below. EI is constant.





$$\sum M = 0$$

$$M + wx(\frac{x}{2}) + Px = 0$$

$$M = -w\frac{x^2}{2} - Px$$

$$\frac{\partial M}{\partial P} = -x$$

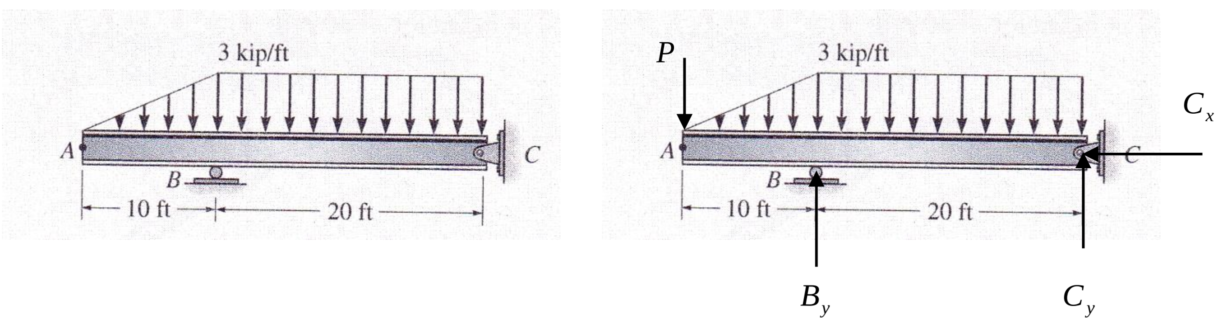
When P=0

$$M = -w\frac{x^2}{2} \;,\; \frac{\partial M}{\partial P} = -x$$

$$\Delta_B = \int_0^L M(\frac{\partial M}{\partial P})\frac{dx}{EI} = \int_0^L \frac{\frac{-wx^2}{2}(-x)}{EI}dx = \frac{w}{2EI}\int_0^L x^3 dx$$

$$\Delta_B = \frac{w}{8EI}x^4\Big|_0^L = \frac{wL^4}{8EI}$$

Example: Determine the displacement of point A of the steel beam shown below. I=450 in^4, E_{st}=29×10^3 ksi.



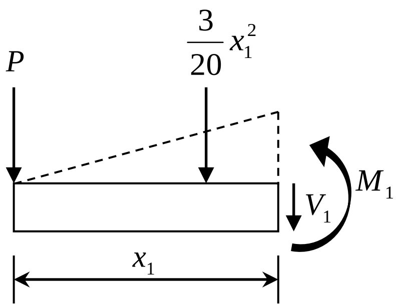
$$\sum M_B = 0$$

C_y×20-60×10+15×$\frac{10}{3}$+P×10=0

C_y =27.5-0.5P

$$\sum F_y = 0$$

B_y+27.5-0.5P-60-15-P=0



B_y=47.5+1.5P

$\sum F_x = 0$

C_x=0

$\sum M = 0$

$M_1 + \frac{3}{20}x_1^2(\frac{x_1}{3}) + Px_1 = 0$

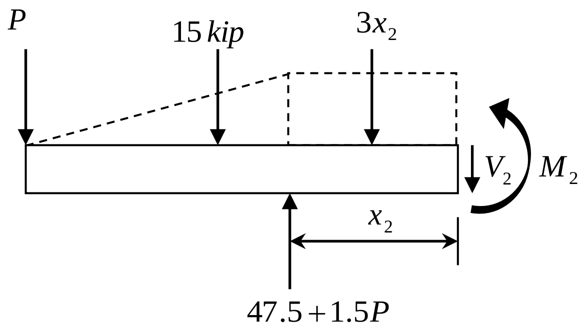
$M_1 = -\frac{x_1^3}{20} - Px_1$

$\frac{\partial M_1}{\partial P} = -x_1$

When P=0

$M_1 = -\frac{x_1^3}{20}$

$\frac{\partial M_1}{\partial P} = -x_1$



$\sum M = 0$

$M_2 + 3x_2(\frac{x_2}{2}) + 15\times(\frac{10}{3} + x_2) + P(10 + x_2) - (47.5 + 1.5P)x_2 = 0$

$M_2 = -\frac{3}{2}x_2^2 + (32.5 + 0.5P)x_2 - 10P - 50$

$\frac{\partial M_2}{\partial P} = 0.5x_2 - 10$

When P=0

$M_2 = -\frac{3}{2}x_2^2 + 32.5x_2 - 50$, $\frac{\partial M_2}{\partial P} = 0.5x_2 - 10$

$$\Delta_A = \int_0^L M_1(\frac{\partial M_1}{\partial P})\frac{dx_1}{EI} + \int_0^L M_2(\frac{\partial M_2}{\partial P})\frac{dx_2}{EI}$$

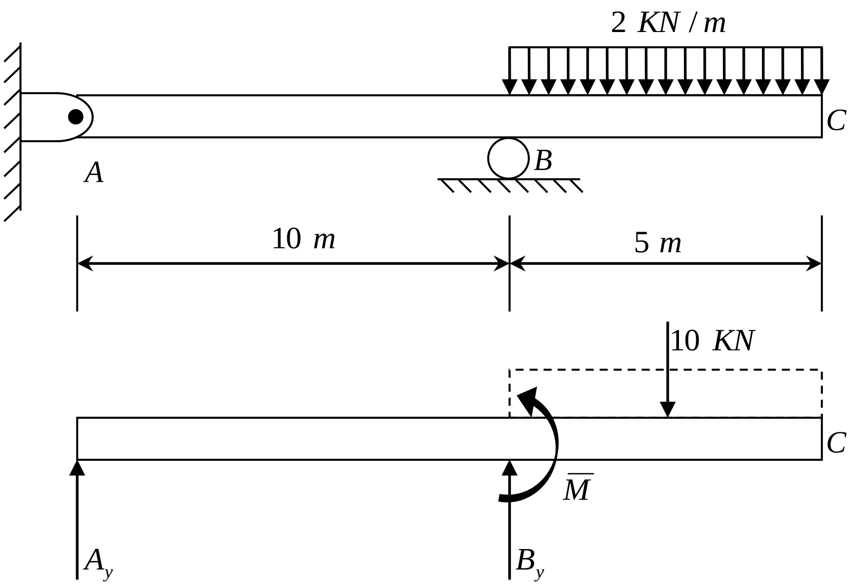
$$\Delta_A = \frac{(12)^3}{EI}\int_0^{10} -\frac{x_1^3}{20}(-x_1)dx_1 + \frac{(12)^3}{EI}\int_0^{20}(-\frac{3}{2}x_2^2 + 32.5x_2 - 50)(0.5x_2 - 10)dx_2$$

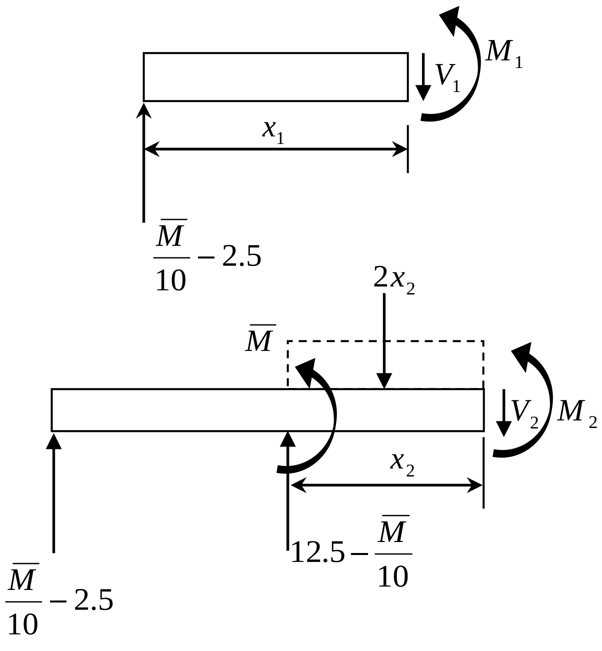
$$\Delta_A = \frac{(12)^3}{EI}\left\{\left|\frac{x_1^5}{100}\right|_0^{10} + \left|\frac{-3}{16}x_2^4 + 10.4166x_2^3 - 175x_2^2 + 500x_2\right|_0^{20}\right\}$$

$$\Delta_A = \frac{(12)^3}{EI}(1000 - 6667.2)$$

$$\Delta_A = -0.75\ in$$

Example: Determine the slope at point B of the A-36 steel beam shown below. $I=70\times10^{6}$ mm^{4} and E=200 GPa.





$\sum M_A = 0$

$B_y\times10+\overline{M}-10\times12.5=0$

$B_y=12.5-\frac{\overline{M}}{10}$

$\sum F_y = 0$

$A_y+12.5-\frac{\overline{M}}{10}-10=0$

$A_y=\frac{\overline{M}}{10}-2.5$

$\sum M = 0$

$M_1-(\frac{\overline{M}}{10}-2.5)x_1=0$

$M_1=(\frac{\overline{M}}{10}-2.5)x_1$

$\frac{\partial M_1}{\partial \overline{M}}=\frac{1}{10}x_1$

When $\overline{M}=0$

$M_1=-2.5x_1$, $\frac{\partial M_1}{\partial \overline{M}}=\frac{1}{10}x_1$

$\sum M = 0$

$M_2+2x_2(\frac{x_2}{2})-(12.5-\frac{\overline{M}}{10})x_2-(\frac{\overline{M}}{10}-2.5)(10+x_2)+\overline{M}=0$

$$M_2=(12.5-\frac{\overline{M}}{10})x_2+(\frac{\overline{M}}{10}-2.5)(10+x_2)-x_2^2-\overline{M}$$

$$\frac{\partial M_2}{\partial \overline{M}}=-\frac{1}{10}x_1+1+\frac{1}{10}x_1-1=0$$

When $\overline{M}=0$

$M_2=-x_2^2+10x_2-25$, $\frac{\partial M_2}{\partial \overline{M}}=0$

$$\theta_B=\int_0^L M(\frac{\partial M}{\partial \overline{M}})\frac{dx}{EI}=\int_0^{10} M_1(\frac{\partial M_1}{\partial \overline{M}})\frac{dx_1}{EI}+\int_0^{5} M_2(\frac{\partial M_2}{\partial \overline{M}})\frac{dx_2}{EI}$$

$$=\int_0^{10}\frac{(-2.5x_1)(\frac{x_1}{10})dx_1}{EI}+0$$

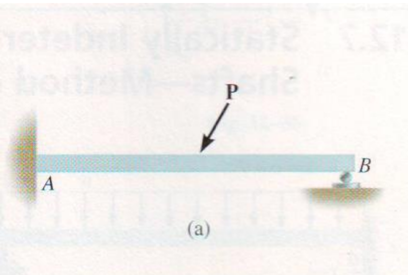
$$=\frac{1}{70\times10^6\times10^{-12}\times200\times10^6}\int_0^{10}-0.25x_1^2dx_1$$

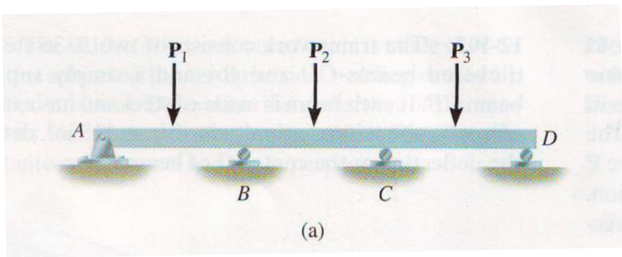
$$=0.07142\times10^{-3}(-0.0833\,x_1^3\Big|_0^{10})$$

$$\theta_B=-0.00595 \quad rad$$

$$=-0.341^0$$

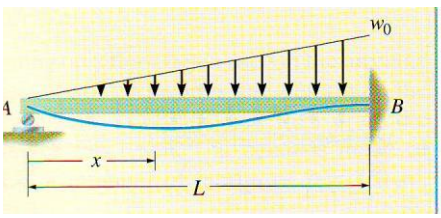
Statically Indeterminate Beams

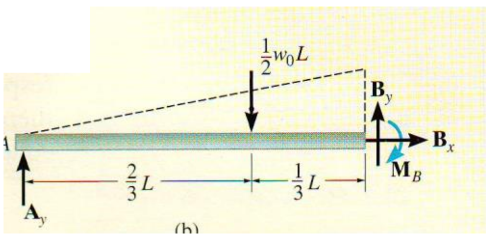




1. **Method of Integration:**

Example: The beam is subjected to the distributed loading shown. Determine the reaction at A. EI is constant.





$$\sum M = 0$$

$$M - A_y x + \frac{w_o x^3}{6L} = 0$$

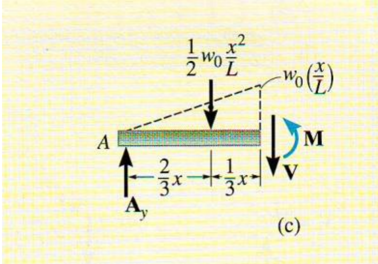
$$M = A_y x - \frac{w_o x^3}{6L}$$

$$M = EI(\frac{d^2 v}{dx^2})$$

$$A_y x - \frac{w_o x^3}{6L} = EI(\frac{d^2 v}{dx^2})$$

$$EI(\frac{dv}{dx}) = \frac{A_y x^2}{2} - \frac{w_o x^4}{24L} + C_1$$

$$EIv = \frac{A_y x^3}{6} - \frac{w_o x^5}{120L} + C_1 x + C_2$$



Boundary Conditions

at x=0 v=0 , at x=L $\frac{dv}{dx} = 0$, at x=L v=0

$0=0-0+0+C_2 \Rightarrow C_2=0$

$$\frac{A_y L^2}{2} - \frac{w_o L^3}{24} + C_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

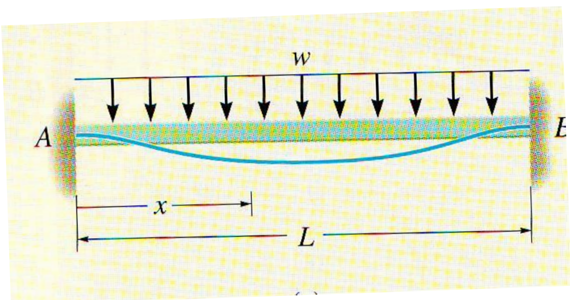
$$\frac{A_y L^3}{6} - \frac{w_o L^4}{120} + C_1 L = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

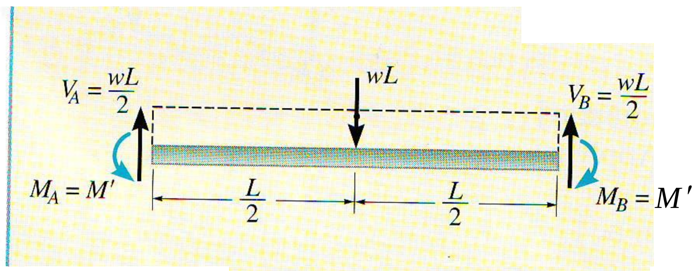
From equations (1) and (2)

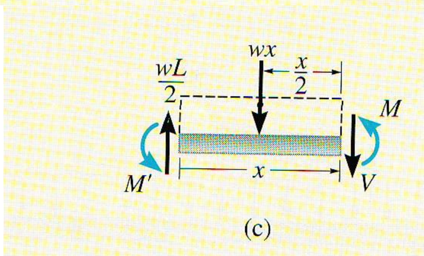
$$C_1 = -\frac{w_o L^3}{24}$$

$$A_y = \frac{w_o L}{10}$$

Example: The beam shown below is fixed supported at both ends and is subjected to the uniform loading shown. Determine the reactions at the supports. Neglect the effect of axial load.







$\sum F_y = 0$

$V_A = V_B = \frac{wL}{2}$

$M_A = M_B = M'$

$\sum M = 0$

$$M + M' + \frac{wx^2}{2} - \frac{wL}{2}x = 0$$

$$M = -\frac{wx^2}{2} + \frac{wL}{2}x - M'$$

$$M = EI(\frac{d^2v}{dx^2})$$

$$EI(\frac{d^2v}{dx^2}) = -\frac{wx^2}{2} + \frac{wL}{2}x - M'$$

$$EI(\frac{dv}{dx}) = -\frac{wx^3}{6} + \frac{wL}{4}x^2 - M'x + C_1$$

$$EIv = -\frac{wx^4}{24} + \frac{wL}{12}x^3 - \frac{M'}{2}x^2 + C_1x + C_2$$

<u>Boundary Conditions</u>

at x=0 v=0 , at x=0 $\frac{dv}{dx} = 0$, at x=L v=0

$0=-0+0-0+0+C_2 \Rightarrow C_2=0$

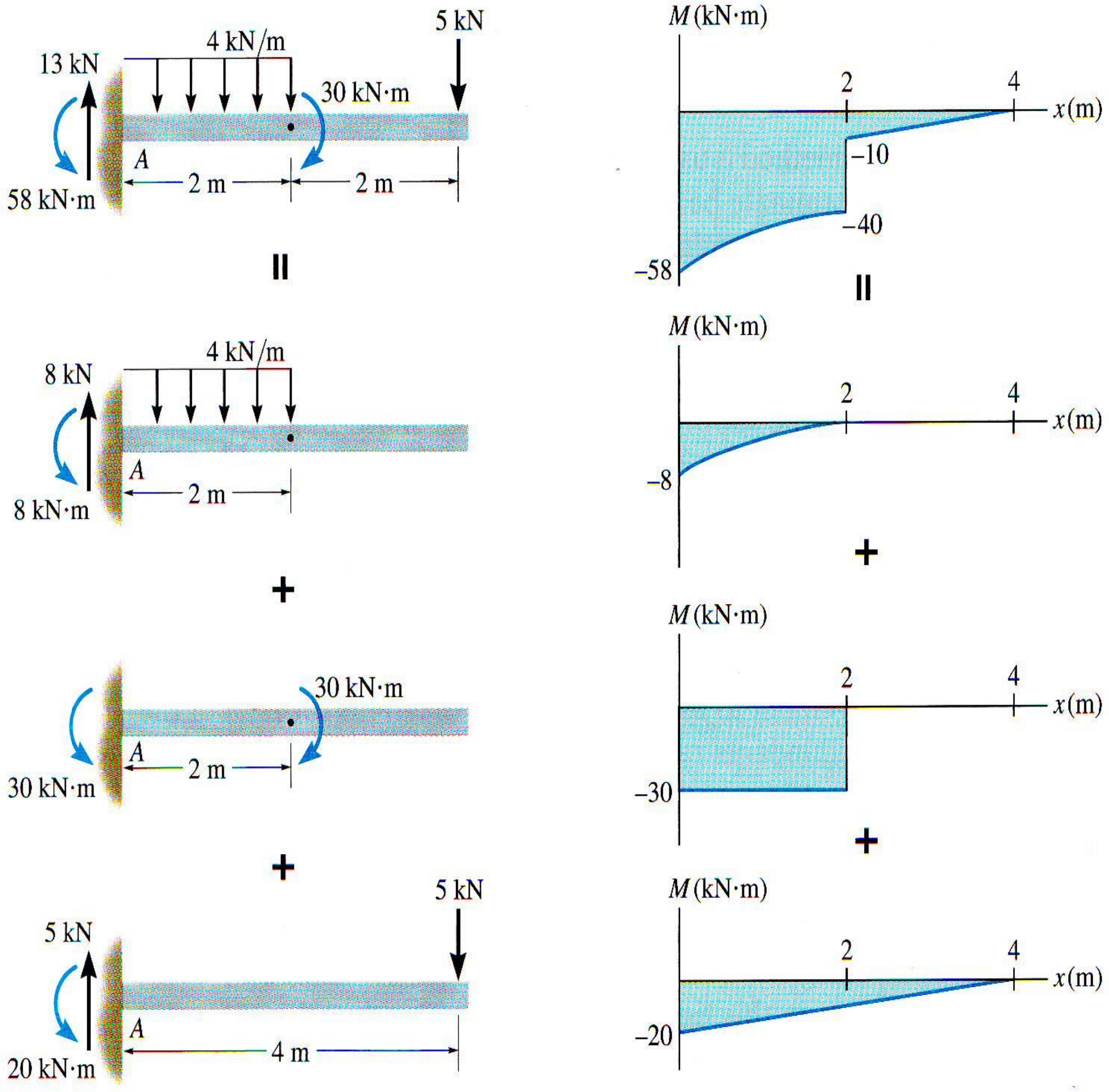
$0=-0+0-0+C_1 \Rightarrow C_1=0$

$$0 = -\frac{wL^4}{24} + \frac{wL^4}{12} - \frac{M'}{2}L^2$$

$$M' = \frac{wL^2}{12}$$

Moment Area Method(Statically Indeterminate):

Since application of the moment area theorems requires calculation of both the area under the M/EI diagram and the centroidal location of this area, it is often convenient to use separate M/EI diagrams for each of the known loads and redundant rather than using the resultant diagram to compute these geometric quantities.



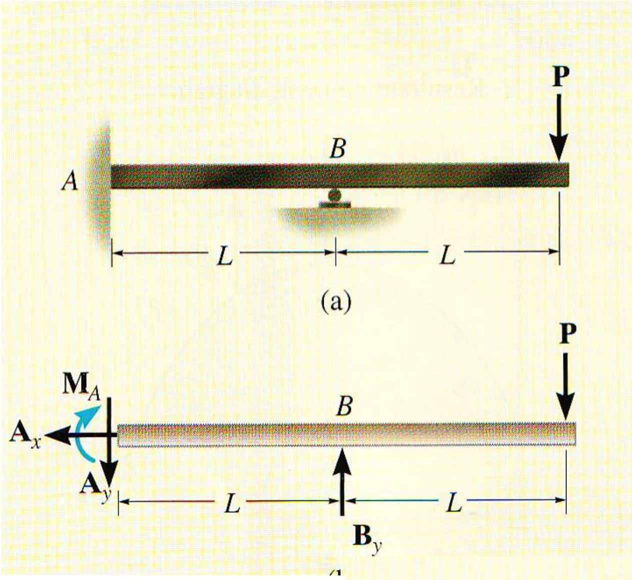
Superposition of loadings

(a)

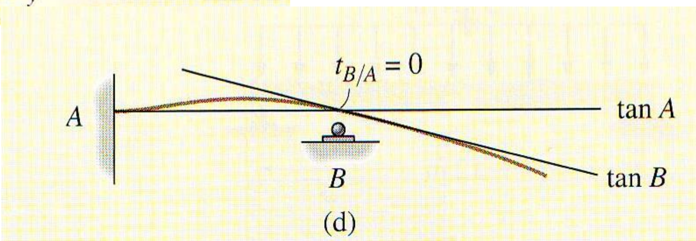
Superposition of moment diagrams

(b)

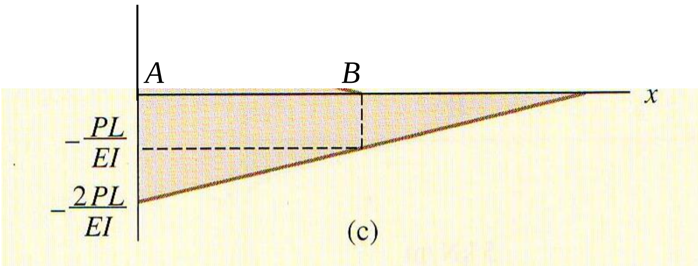
Example: The beam is subjected to the concentrated loading shown. Determine the reactions of the supports. EI is constant.

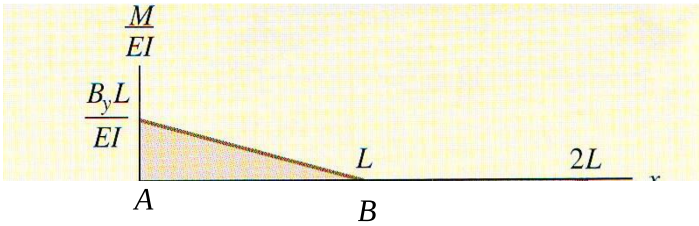


From the elastic curve $\Delta_B = 0$, $t_{B/A}=0$



Using superposition method to draw the separate M/EI diagrams for the redundant reaction B_y and the load P.





For load P

For redundant reaction B_y

$$\frac{1}{2}(\frac{B_yL}{EI})(L)(\frac{2}{3}L)+(-\frac{PL}{EI})(L)(\frac{L}{2})+\frac{1}{2}(-\frac{PL}{EI})(L)(\frac{2}{3}L)=0$$

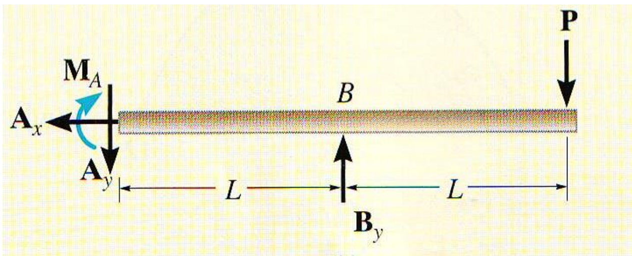
$$B_y = 2.5P$$

$$\sum F_y = 0$$

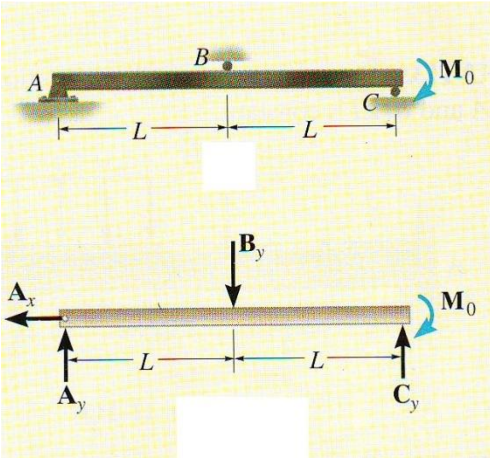
$-A_y-P+2.5P=0 \Rightarrow A_y=1.5P$

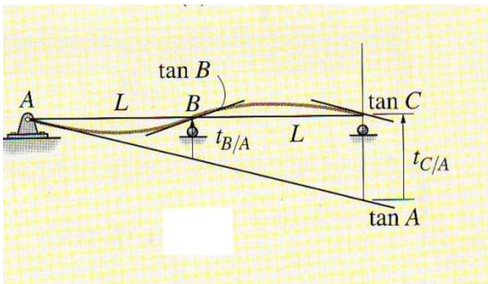
$$\sum M_A = 0$$

$M_A=0.5PL$



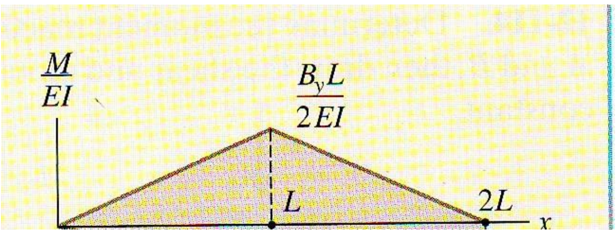
Example: The beam is subjected to the couple moment at it end C as shown below. Determine the reaction at B. EI is constant.

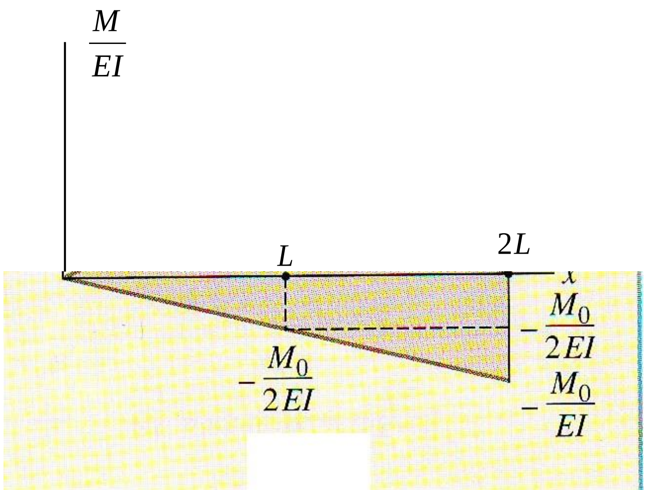




From the elastic curve $\frac{t_{C/A}}{2L} = \frac{t_{B/A}}{L} \Rightarrow t_{C/A} = 2t_{B/A}$

Using superposition method to draw the separate M/EI diagrams for the redundant reaction B_y and the load M_o.





For redundant reaction B_y For load M_o

$$t_{B/A} = (\frac{1}{3}L)[\frac{1}{2}(L)(\frac{B_y L}{2EI})] + (\frac{1}{3}L)[\frac{1}{2}(L)(-\frac{M_o}{2EI})] = \frac{L^3}{12EI}(B_y - \frac{M_o}{L})$$

$$t_{C/A} = (L + \frac{1}{3}L)[\frac{1}{2}(L)(\frac{B_y L}{2EI})] + (\frac{2}{3}L)[\frac{1}{2}(L)(\frac{B_y L}{2EI})] + (\frac{2}{3}L)[\frac{1}{2}(2L)(-\frac{M_o}{EI})] = \frac{L^3}{12EI}(6B_y - 8\frac{M_o}{L})$$

$$\frac{L^3}{12EI}(6B_y - 8\frac{M_o}{L}) = 2\frac{L^3}{12EI}(B_y - \frac{M_o}{L}) \Longrightarrow B_y = \frac{3}{2}\frac{M_o}{L}$$

$$A_y = \frac{M_o}{4L}, \quad C_y = \frac{5M_o}{L}$$

Combined Stresses

There are three types of loading: axial, torsional and flexural.

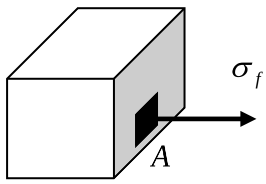
Axial loading $\sigma_a = \dfrac{P}{A}$

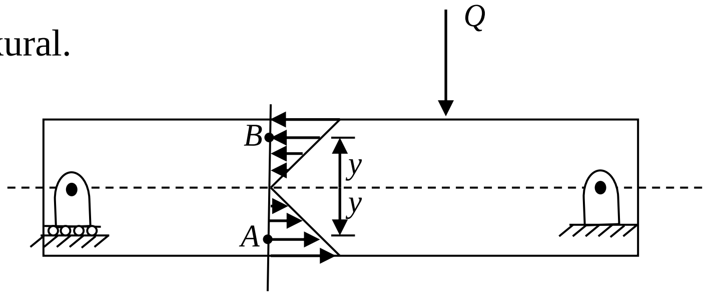
Torsional loading $\tau = \dfrac{T.r}{J}$

Flexural loading $\sigma_f = \dfrac{M.y}{I}$

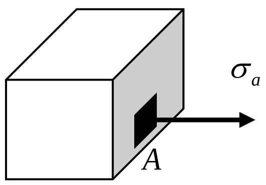
There are four possible combinations of these loadings:

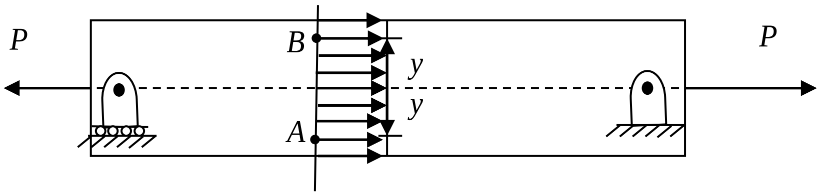
1. Axial and flexural.
2. Axial and torsional.
3. Torsional and flexural.
4. Axial , torsional and flexural.



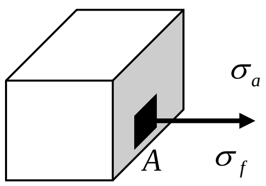


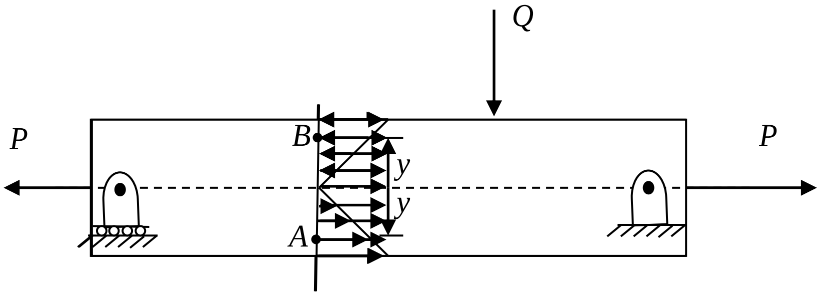
$$\sigma_f = \frac{M.y}{I}$$





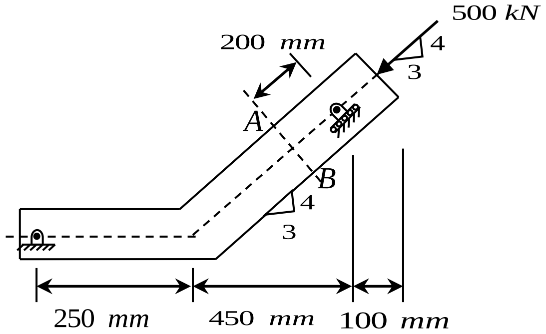
$$\sigma_a = \frac{P}{A}$$





For point A $\sigma_A = \sigma_a + \sigma_f$ **for point B** $\sigma_A = \sigma_a - \sigma_f$

Example: The bent steel bar shown is ***200 mm*** square. Determine the normal stresses at ***A*** and ***B***.



$\sum M_C = 0$

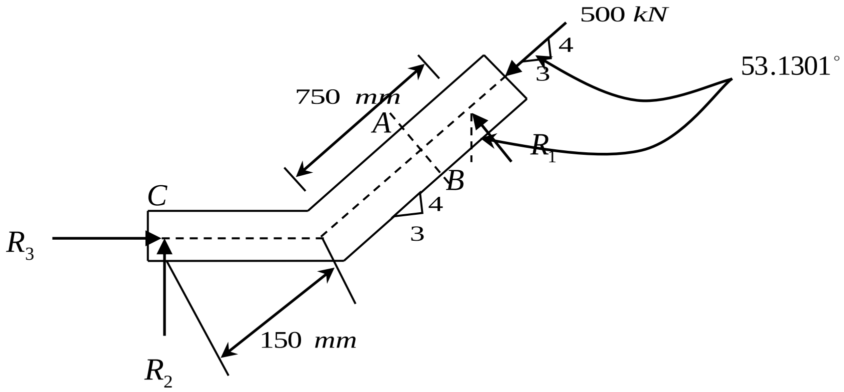
$-500\times200\times10^{-3}+R_1\times900\times10^{-3}=0$

$R_1=111.111$ kN

$\sum F_y = 0$

$R_2+111.111\cos(53.1301)-500\sin(53.1301)=0$

$R_2=333.333$kN



$\sum F_x = 0$

$R_3+111.111\sin(53.1301)-500\cos(53.1301)=0$

$R_3=388.888$kN

$$\sigma_A = -\frac{P}{A} = -\frac{500}{(200\times200\times10^{-6})} = -12.5\,MPa$$

$M=-500\times200\times10^{-3}+111.111\times700\times10^{-3}$

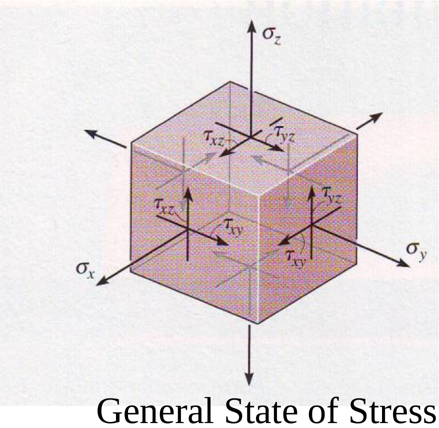
$M=-22.2223$ kN.m

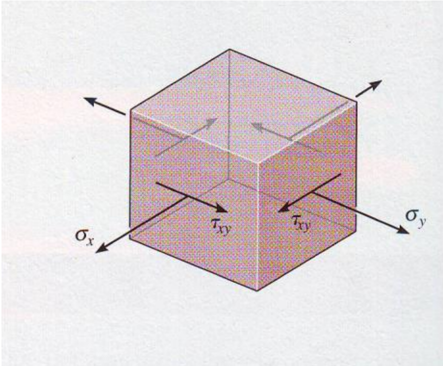
$$\sigma_f = \frac{M.y}{I} = \frac{22.2223\times100\times10^{-3}}{(200\times10^{-3})\ \times(200\times10^{-3})^3/12} = 16.666\,MPa$$

$$\sigma_A = -\sigma_a - \sigma_f = -12.5-16.666 = -29.166\ MPa$$

$$\sigma_B = -\sigma_a + \sigma_f = -12.5+16.666 = 4.166\ MPa$$

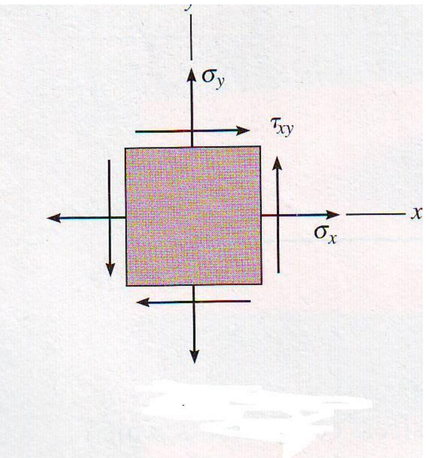
Stresses at a Point

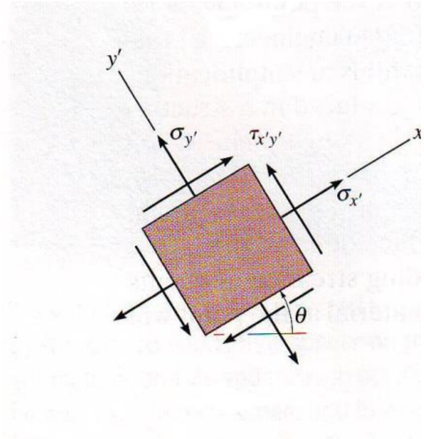




General State of Stress

Plane Stress





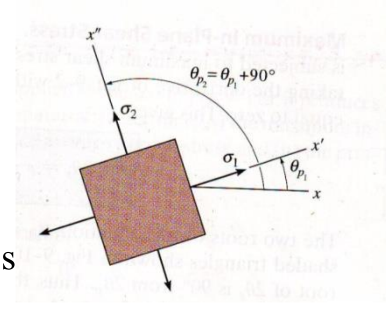
$$\sigma_{\bar{x}} = (\frac{\sigma_x + \sigma_y}{2}) + (\frac{\sigma_x - \sigma_y}{2})\cos 2\theta + \tau_{xy}\sin 2\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

$$\sigma_{\bar{y}} = (\frac{\sigma_x + \sigma_y}{2}) - (\frac{\sigma_x - \sigma_y}{2})\cos 2\theta - \tau_{xy}\sin 2\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\tau_{\bar{x}\bar{y}} = -(\frac{\sigma_x - \sigma_y}{2})\sin 2\theta + \tau_{xy}\cos 2\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The planes defining maximum or minimum normal stresses are found from:

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$



The planes of maximum shearing stresses are defined by :

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

The planes of zero shearing stresses may be determined by s

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Equation 6 and 4 show that maximum and minimum normal stresses occur on planes of zero shearing stresses.

The maximum and minimum normal stresses are called the ***principal stresses.***

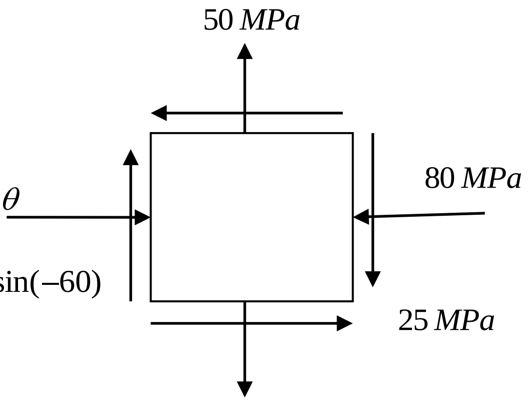
Equation 5 is the negative reciprocal of equation 4. This means that the values of $2\theta_s$ from equation 5 and equation 4 differ by 90^o. This means that the planes of maximum shearing stress are at 45^o with the planes of principal stress.

$$\sigma_{\substack{max. \\ min.}} = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(8)$$

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Example: The state of plane stress at a point is represented by the element shown. Determine the state of stress at the point on another element oriented 30^o clockwise from the position shown.



$\sigma_x = -80\ MPa$

$\sigma_y = 50\ MPa$

$\tau_{xy} = -25\ MPa$

$\theta = -30^o$

$$\sigma_{\bar{x}} = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{\bar{x}} = \left(\frac{-80+50}{2}\right) + \left(\frac{-80-50}{2}\right)\cos(-60) - 25\sin(-60)$$

$$\sigma_{\bar{x}} = -25.849\ MPa$$

$$\sigma_{\bar{y}} = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta - \tau_{xy}\sin 2\theta$$

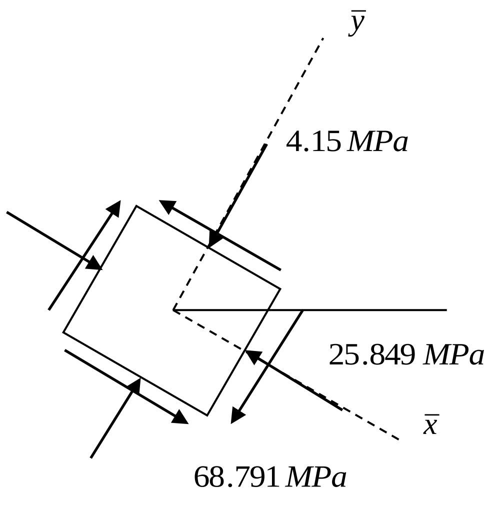
$$\sigma_{\bar{y}} = \left(\frac{-80+50}{2}\right) - \left(\frac{-80-50}{2}\right)\cos(-60) + 25\sin(-60)$$

$$\sigma_{\bar{y}} = -4.15\ MPa$$

$$\tau_{\bar{x}\bar{y}} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{\bar{x}\bar{y}} = -\left(\frac{-80-50}{2}\right)\sin(-60) - 25\cos(-60)$$

$$\tau_{\bar{x}\bar{y}} = -68.791\ MPa$$

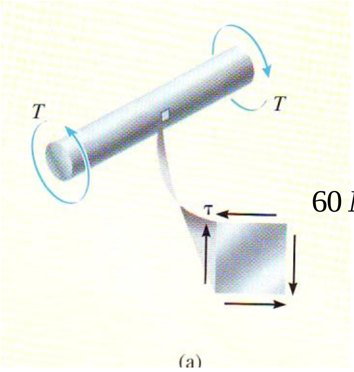


Example: When the torsional loading T is applied to the bar shown it produce a state of pure shear stress in the material. Determine a) the maximum in plane shear stress and the associated average normal stress. b) the principal stresses.

$\sigma_x = 0$

$\sigma_y = 0$

$\tau_{xy} = -60\,MPa$



a) $\tau_{\max} = \pm\sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}{}^2}$

$$\tau_{\max} = \pm\sqrt{(\frac{0-0}{2})^2 + (-60)^2} = 60\,MPa$$

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{0+0}{2} = 0$$

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\tan 2\theta_s = -\frac{0}{2\times -60} = 0$$

$\theta_s=0$

b) $\sigma_{\substack{\max.\\ \min.}} = (\frac{\sigma_x + \sigma_y}{2}) \pm \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}{}^2}$

$$\sigma_{\substack{\max.\\ \min.}} = (\frac{0+0}{2}) \pm \sqrt{(\frac{0-0}{2})^2 + (-60)^2} = \pm 60\,MPa$$

σ_{max}=60 MPa

σ_{min}=-60 MPa

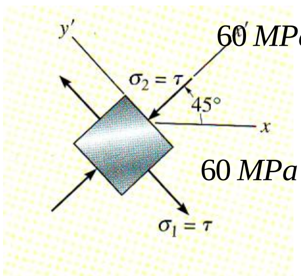
$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\tan 2\theta_p = \frac{2\times -60}{0} = \infty$$

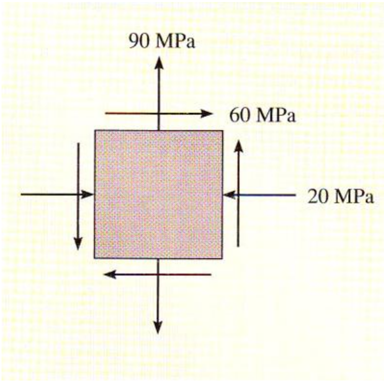
$$2\theta_p = \frac{\pi}{2} \longrightarrow \theta_p = \frac{\pi}{4} = 45^\circ \text{ or } 135^\circ$$

$$\sigma_{\bar{x}} = (\frac{\sigma_x + \sigma_y}{2}) + (\frac{\sigma_x - \sigma_y}{2})\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{\bar{x}} = (\frac{0}{2}) + (\frac{0}{2})\cos(90) - 60\sin(90) = -60\,MPa$$



Example: The state of plane stress at a point on a body is shown on the element. Represent this stress state in terms of the principal stresses.



$$\sigma_x = -20\,MPa$$

$$\sigma_y = 90\,MPa$$

$$\tau_{xy} = 60\,MPa$$

$$\sigma_{\substack{max.\\min.}} = (\frac{\sigma_x + \sigma_y}{2}) \pm \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}{}^2}$$

$$\sigma_{\substack{max.\\min.}} = (\frac{-20+90}{2}) \pm \sqrt{(\frac{-20-90}{2})^2 + (60)^2}$$

$$\sigma_{\substack{max.\\min.}} = 35 \pm 81.394$$

σ_{max}=116.394 MPa

σ_{min}=-46.394 MPa

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2 \times 60}{-20-90} = -1.090909$$

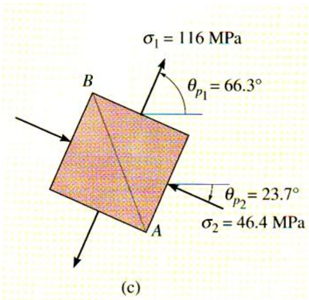
$$2\theta_p = -47.489 \longrightarrow \theta_p = -23.744 \text{ or } 66.256^o$$

$$\sigma_{\bar{x}} = (\frac{\sigma_x + \sigma_y}{2}) + (\frac{\sigma_x - \sigma_y}{2})\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{\bar{x}} = (\frac{-20+90}{2}) + (\frac{-20-90}{2})\cos(-47.489) + 60\sin(-47.489) = -46.349\,MPa$$

θ_{p1}=66.256^o

θ_{p2}=-23.744^o



(c)

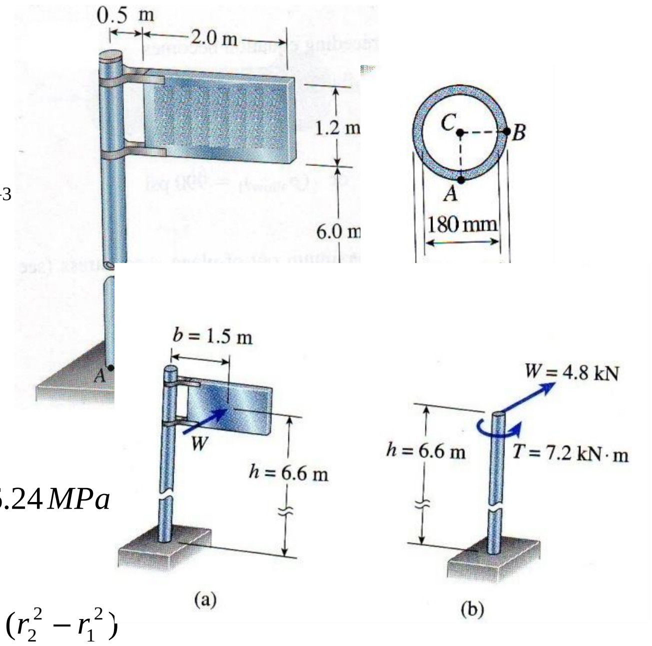
Example: A sign of dimensions 2 m×1.2 m is supported by a hollow circular pole having outer diameter 220 mm and inner diameter 180 mm as shown below. The sign is offset 0.5 m from the center line of the pole and its lower edge is 6 m above the ground. Determine the principal stresses and maximum shear stresses at points A and B at the base of the pole due to wind pressure of 20 kPa against the sign.

w=PA=2×(2×1.2)=4.8 kN

T=wr=4.8×(1+0.5)=7.2 kN.m

M=wd=4.8×(6+0.6)=31.68 kN.m

V=w=4.8 kN



$$I=\frac{\pi}{64}[d_2^4-d_1^4]=\frac{\pi}{64}[(220\times10^{-3})^4-(180\times10^{-3}$$

I=63.46×10^{-6} m^4

$$\sigma_A=\frac{M.y}{I}=\frac{31.68\times\frac{220\times10^{-3}}{2}}{36.46\times10^{-6}}=54.91\ MPa$$

$$\tau_1=\frac{T.r}{J}=\frac{7.2\times\frac{220\times10^{-3}}{2}}{\frac{\pi}{32}[(220\times10^{-3})^4-(180\times10^{-3})^4]}=6.24\,MPa$$

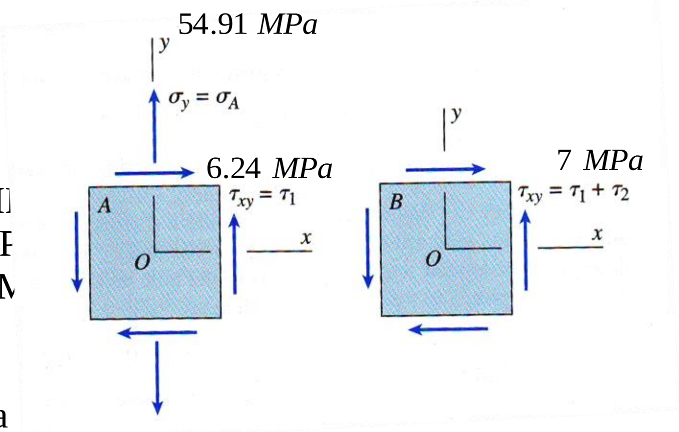
$$\tau_2=\frac{VQ}{It}\ ,t=2(r_2-r_1)\ ,I=\frac{\pi}{4}(r_2^4-r_1^4)\ \ ,\bar{A}=\frac{\pi}{2}(r_2^2-r_1^2)$$

$$\bar{y}=\frac{\bar{y}_1A_1-\bar{y}_2A_2}{A_1-A_2}=\frac{(\frac{4r_2}{3\pi})(\frac{\pi}{2}r_2^2)-(\frac{4r_1}{3\pi})(\frac{\pi}{2}r_1^2)}{\frac{\pi}{2}r_2^2-\frac{\pi}{2}r_1^2}=\frac{4}{3\pi}[\frac{r_2^3-r_1^3}{r_2^2-r_1^2}]$$

$$Q=\bar{y}\bar{A}=\frac{4}{3\pi}[\frac{r_2^3-r_1^3}{r_2^2-r_1^2}]\times\frac{\pi}{2}(r_2^2-r_1^2)=\frac{2}{3}(r_2^3-r_1^3)$$

$$\tau_2=\frac{V\frac{2}{3}(r_2^3-r_1^3)}{\frac{\pi}{4}(r_2^4-r_1^4)\times2(r_2-r_1)}=\frac{4V(r_2^2+r_2r_1+r_1^2)}{3\pi(r_2^2-r_1^2)(r_2^2+r_1^2)}$$

τ_2=0.76 MPa



Point A

σ_1=55.7 M

σ_2=-0.7 MP

τ_{max}=28.2 M

Point B

σ_1=7 MPa

σ_2=-7 MPa

τ_{max}=7 MPa

Mohr's Circle

For plane stresses transformation have a graphical solution that is often convenient to use and easy to remember. Furthermore this approach will allow us to visualize how the normal and shear stress components $\sigma_{\bar{x}}$ and $\tau_{\bar{x}\bar{y}}$ vary as the plane on which they act is oriented in different directions. This graphical solution known as Mohr's circle.

$$\sigma_{\bar{x}} = (\frac{\sigma_x + \sigma_y}{2}) + (\frac{\sigma_x - \sigma_y}{2})\cos 2\theta + \tau_{xy}\sin 2\theta$$

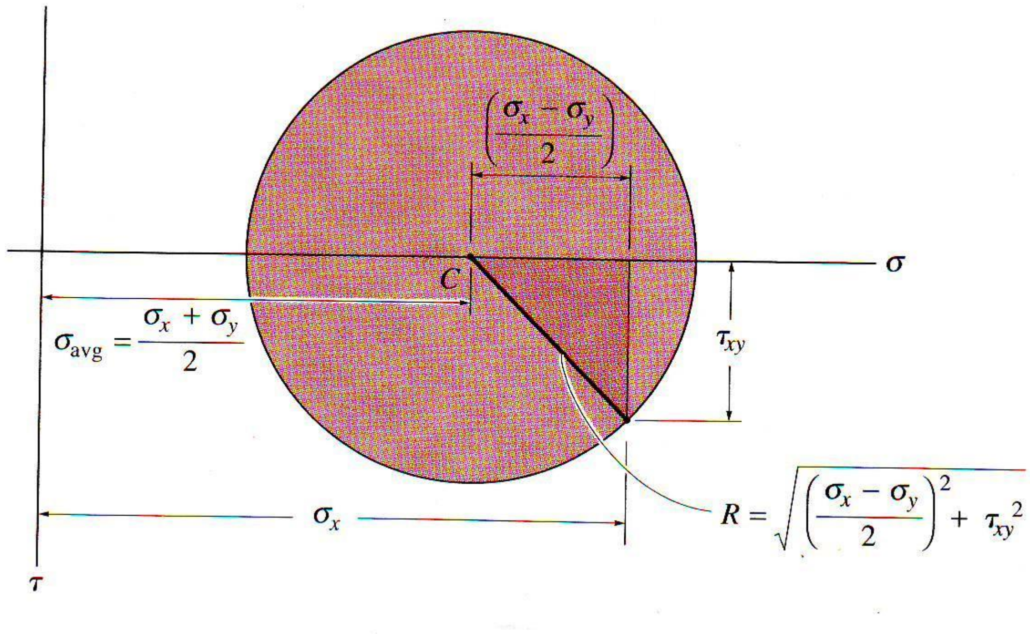
$$\tau_{\bar{x}\bar{y}} = -(\frac{\sigma_x - \sigma_y}{2})\sin 2\theta + \tau_{xy}\cos 2\theta$$

The parameter θ can be eliminated by squaring each equation and adding the equations together. The result is:

$$[\sigma_{\bar{x}} - (\frac{\sigma_x + \sigma_y}{2})]^2 + \tau_{\bar{x}\bar{y}}{}^2 = (\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}{}^2$$

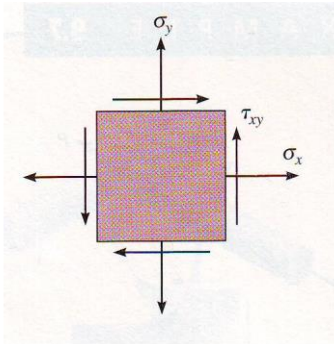
Let $c = \frac{\sigma_x + \sigma_y}{2}$, $R^2 = (\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}{}^2$

$[\sigma_{\bar{x}} - c]^2 + \tau_{\bar{x}\bar{y}}{}^2 = R^2$ this equation represents a circle having a radius R and center at point (c,0).



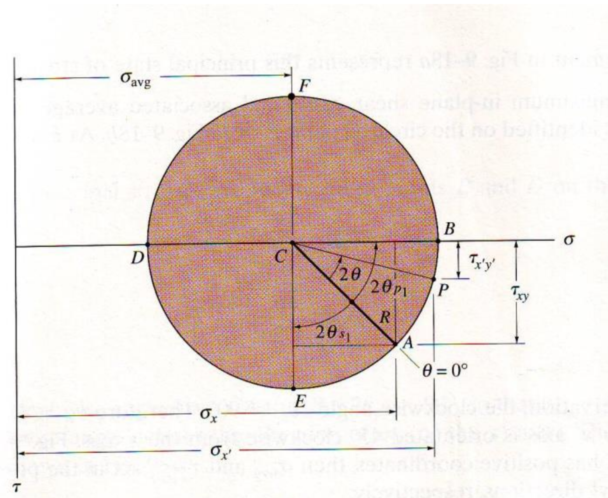
Construction of the circle

1. Establish a coordinate system such that the abscissa represents the normal stress σ with positive to the right and the ordinate represents the shear stress τ with positive down ward.
2. Using the positive sign convention for $\sigma_x \sigma_y \tau_{xy}$ as shown:



Plot the center of the circle C which is located on the σ axis at a distance $\sigma_{avg} = \dfrac{\sigma_x + \sigma_y}{2}$ from the origin.

3. Plot the reference point A having coordinate A(σ_x, τ_{xy}). This point represents the normal and shear stress components on the element's right hand vertical face, and since the $\bar{x}$ axis coincides with the x axis, this represents θ=0.
4. Connect point A with the center C of the circle and determine CA by trigonometry. This distance represents the radius R of the circle.
5. Once R has been determined , sketch the circle.

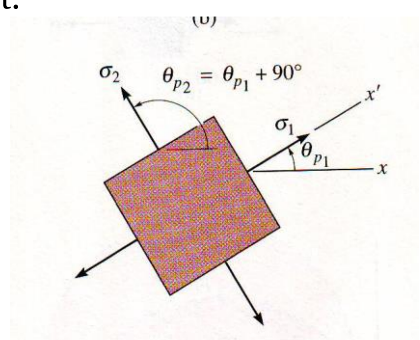


Principal Stresses

The principal stresses σ_1 and σ_2 ($\sigma_1 \geq \sigma_2$) are represented by the two points B and D where the circle intersects the σ axis i.e where τ=0.

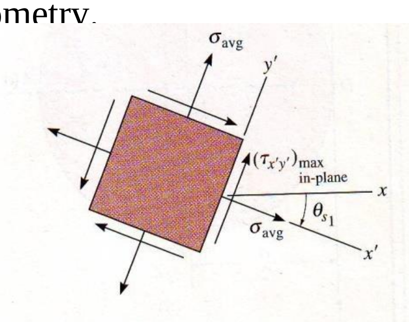
These stresses act on planes defined by angles θ_{p1} and θ_{p2}. They are represented on the circle by angles $2\theta_{p1}$ and $2\theta_{p2}$and are measured from the radial reference line CA to line CB and CD respectively.

Using trigonometry only one of these angles needs to be calculated from the circle since θ_{p1} and θ_{p2} are 90° apart.



Maximum in Plane Shear Stress.

The average normal stress and maximum in plane shear stress components are determined from the circle as the coordinates of either point E or F. In this case the angles θ_{s1} and θ_{s2} give the orientation of the planes that contain these components. The angle $2\theta_{s1}$ can be determined using trigonometry.



Stress on Arbitrary Plane.

The normal and shear stress components $\sigma_{\bar{x}}$ and $\tau_{\bar{x}\bar{y}}$ acting on a specified plane defined by the angle θ can be obtained from the circle using trigonometry to determine the coordinates of point P.

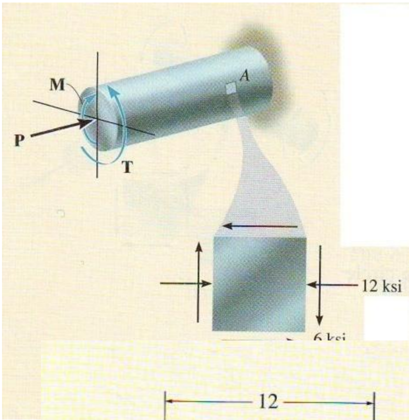
To locate P, the known angle θ for the plane must be measured on the circle in the same direction 2θ from the radial reference line CA to the radial line CP.

Example: Due to the applied loading the element at point A on the solid cylinder is subjected to the state of stress shown. Determine the principal stresses acting at this point.

$\sigma_x = -12$ ksi

$\sigma_y = 0$

$\tau_{xy} = -6$ ksi



$$c = \frac{\sigma_x + \sigma_y}{2} = \frac{-12+0}{2} = -6 \text{ ksi}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

$$R = \sqrt{\left(\frac{-12-0}{2}\right)^2 + (-6)^2} = 8.485 \text{ ksi}$$

$\sigma_1 = c + R$

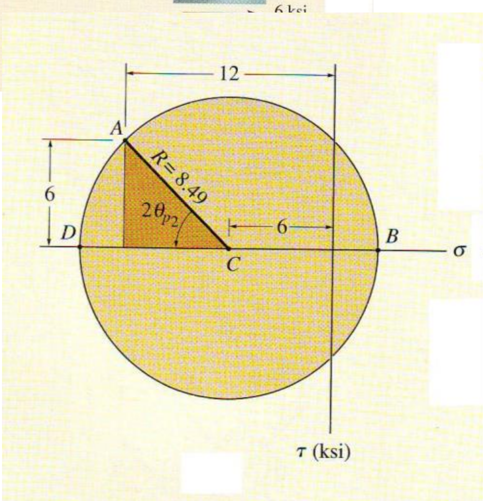
$\sigma_1 = -6+8.485=2.485$ ksi

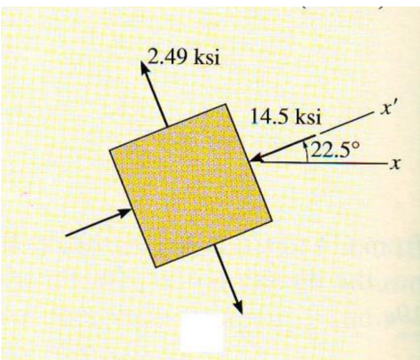
$\sigma_2 = c - R$

$\sigma_2 = -6-8.485=-14.485$ ksi

$$2\theta_{p2} = \tan^{-1}\left(\frac{6}{12-6}\right) = \tan^{-1}(1) = 45^{o}$$

$\theta_{p2} = 22.5^{o}$



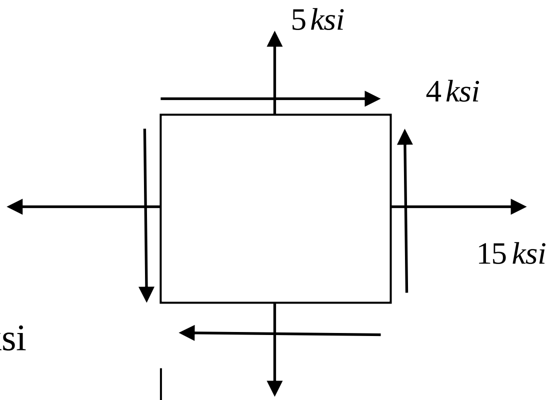


Example: An element in plane stress at the surface of a large machine is subjected to stresses shown below. Using Mohr's circle determine the following quantities a) the stress acting on element inclined at an angle 40° b)the principal stresses and c) the maximum shear stress.

$\sigma_x = 15\text{ksi}$

$\sigma_y = 5 \text{ ksi}$

$\tau_{xy} = 4\text{ksi}$

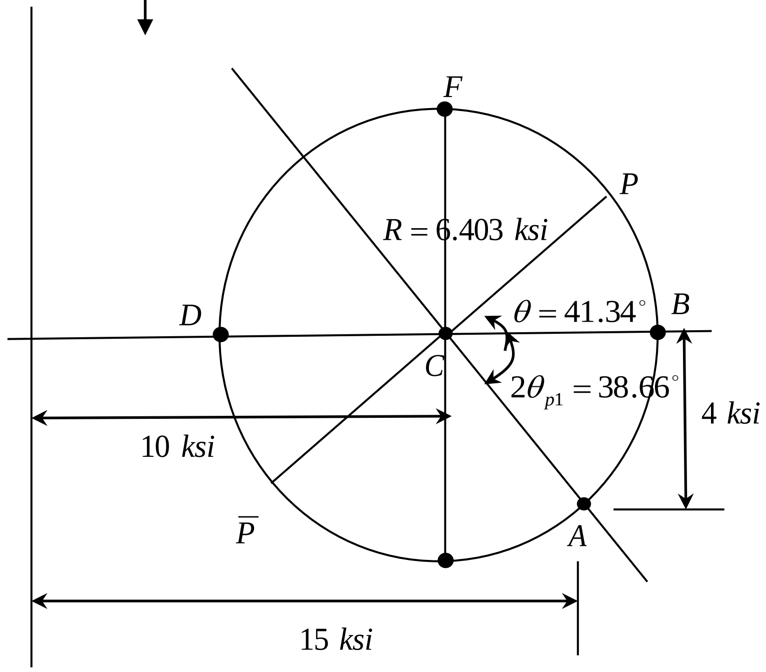


$$c = \frac{\sigma_x + \sigma_y}{2} = \frac{15+5}{2} = 10\,\text{ksi}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

$$R = \sqrt{\left(\frac{15-5}{2}\right)^2 + (4)^2} = 6.403 \text{ ksi}$$

A(15,4)



$$2\theta_{p1} = \sin^{-1}\left(\frac{4}{6.403}\right) = 38.66^{o}$$

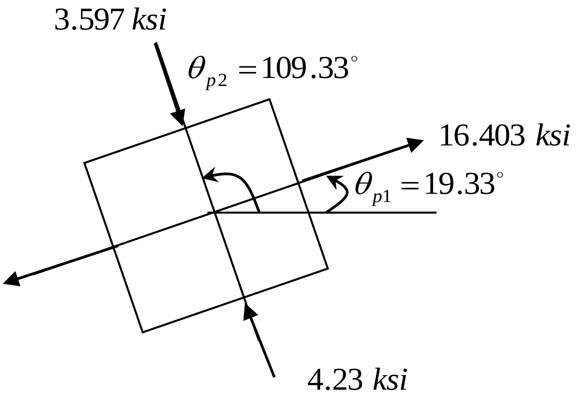
$\theta_{p1} = 19.33^\circ$

$\sigma_1 = c + R$

$\sigma_1 = 10+6.403=16.403 \text{ ksi}$

$\sigma_2 = c - R$

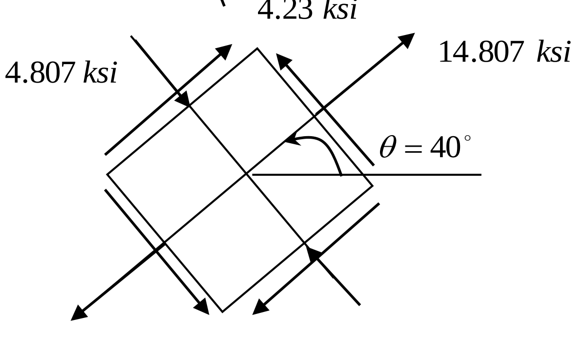
$\sigma_2 = 10\text{-}8.485=3.597\text{ksi}$



$\sigma_{\bar{x}} = 10 + 6.403\cos(41.34) = 14.807 \ ksi$

$\sigma_{\bar{y}} = 10 - 6.403\cos(41.34) = 4.807 \ ksi$

$\tau_{\bar{x}\bar{y}} = 6.403\sin(41.34) = 4.23 \ ksi$

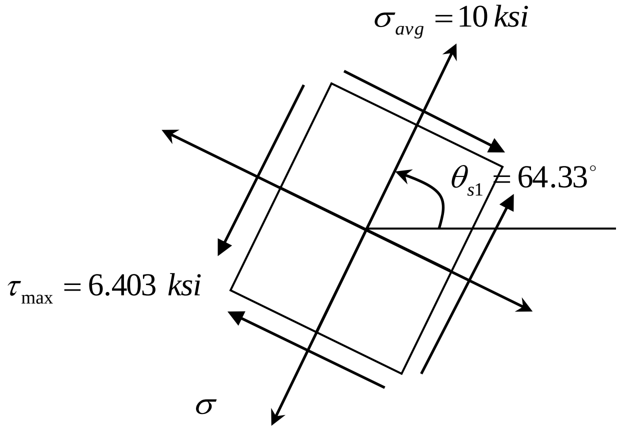


$\tau_{\max} = R = 6.403\ ksi$

$\sigma_{avg} = c = 10\, ksi$

$2\theta_{s1} = 38.66 + 90 = 128.66^{o}$

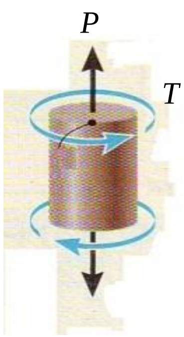
$\theta_{s1} = 64.33^{o}$ counterclockwise

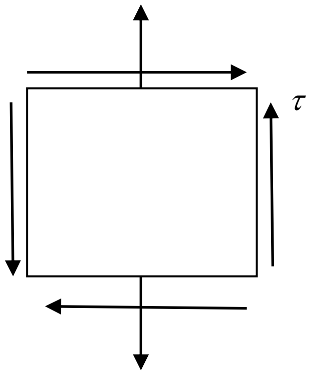


Stresses Due to Axial Load and Torsion

$$\sigma = \frac{P}{A}$$

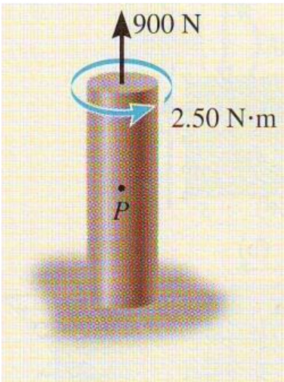
$$\tau = \frac{T\,c}{J}$$

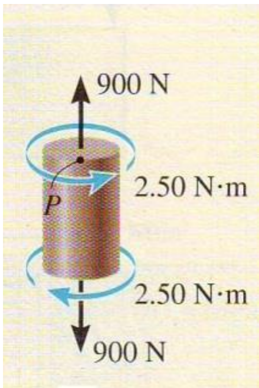




σ_1, σ_2 from Mohr's circle or from stress transformation equations.

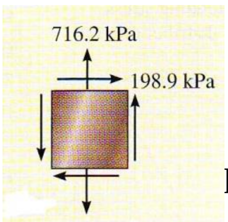
Example: An axial force of 900 N and a torque of 2.5 N.m are applied to the shaft as shown. If the shaft has a diameter of 40 mm, determine the principal stresses at appoint P on its surface.





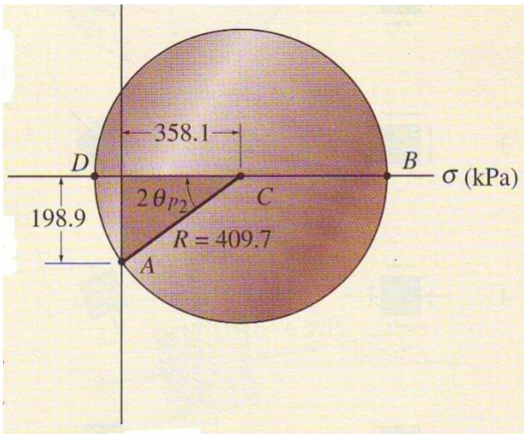
$$\tau = \frac{T\,r}{J} = \frac{2.5 \times (20 \times 10^{-3})}{\frac{\pi}{2}(20 \times 10^{-3})^4} = 198.94367 \text{ kPa}$$

$$\sigma = \frac{P}{A} = \frac{900}{\pi (20 \times 10^{-3})^2} = 716.19724 \text{ kPa}$$

$\sigma_x = 0$, σ_y  Pa , $\tau = 198.94367$ kPa

$$c=\frac{\sigma_x+\sigma_y}{2}=\frac{716.19724}{2}=358.09862\ kPa$$

$$R=\sqrt{(\frac{\sigma_x-\sigma_y}{2})^2+\tau_{xy}{}^2}=\sqrt{(\frac{0-716.19724}{2})^2+(198.94367)^2}=409.65\ kPa$$

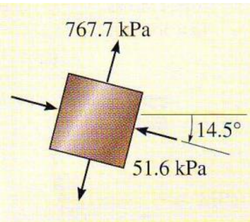


$$\sigma_1=c+R=358.09862+409.65=767.74862\ \text{kPa}$$

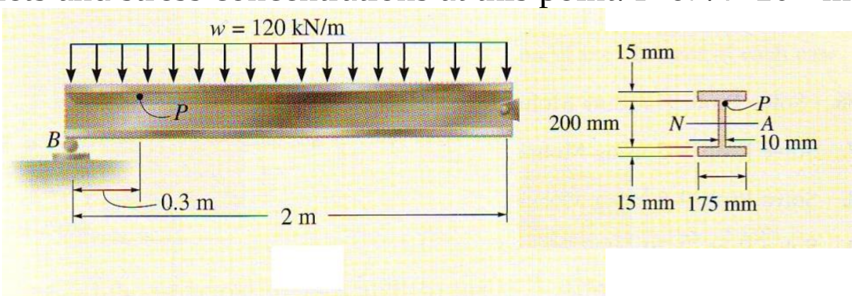
$$\sigma_2=c-R=358.09862-409.65=-51.55138\ \text{kPa}$$

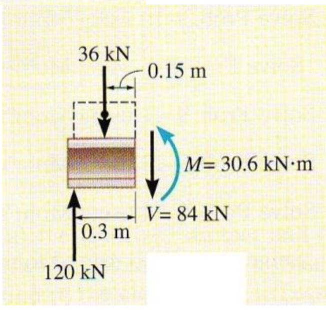
$$2\theta_{p2}=\sin^{-1}(\frac{198.94367}{409.65})=29.054$$

$$\theta_{p2}=14.527\ \text{clockwise}$$



Example: The beam shown below is subjected to the distributed loading of w=120 kN/m. Determine the principal stresses in the beam at point P, which lies at the top of the web. Neglect the size of the fillets and stress concentrations at this point. $I=67.4\times10^{-6}\ m^4$.





$A_x=0$

$\sum M_B=0$

$A_y\times2-240\times1=0 \implies A_y=120$ kN

$B_y=120$ kN

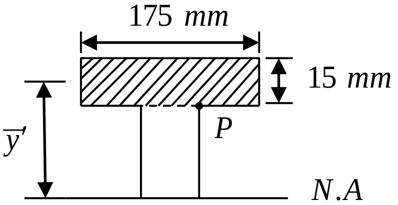
$-V-36+120=0$

$V=84$ kN

M-120×0.3+36×0.15=0

M=30.6 kN.m

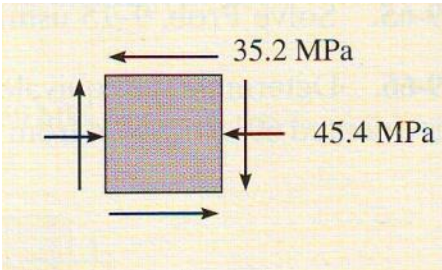
$$\sigma = -\frac{My}{I} = -\frac{30.6\times10^{3}\times100\times10^{-3}}{67.4\times10^{-6}} = -45.4\,MPa$$



$$\tau = \frac{VQ}{It}$$

$Q = \overline{y}'A'$ =(107.5×10^{-3})×(175×10^{-3}×15×10^{-3})=0.000282187 m^3

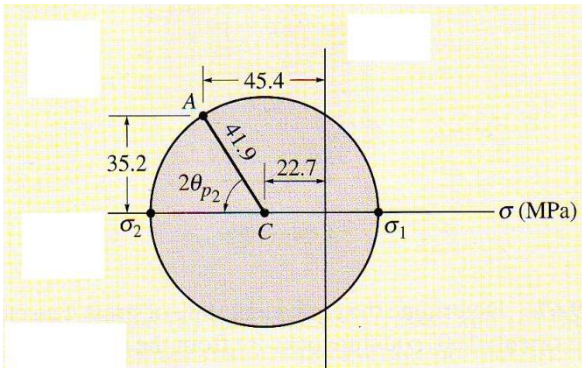
$$\tau = \frac{84\times0.000282187}{67.4\times10^{-6}\times10\times10^{-3}} = 35.168\ MPa$$



$\sigma_x = -45.4\ MPa\ \ \sigma_y = 0\ \ \tau = -35.168\ MPa$

$$c = \frac{\sigma_x + \sigma_y}{2} = \frac{-45.4}{2} = -22.7\ \ MPa$$

$$R = \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}{}^2} = \sqrt{(\frac{0-45.4}{2})^2 + (-35.168)^2} = 41.857\ MPa$$

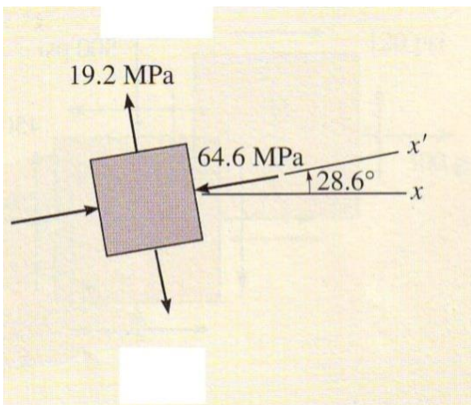


$\sigma_1 = c + R = -22.7 + 41.857$ =19.157 MPa

$\sigma_2 = c - R = -22.7 - 41.857$ =-64.557 MPa

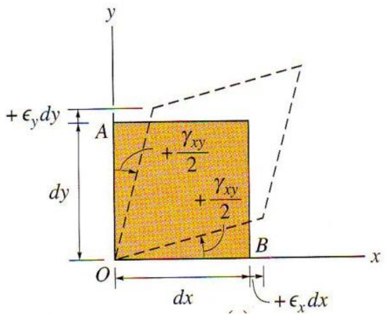
$$2\theta_{p2} = \sin^{-1}(\frac{35.168}{41.857}) = 57.16$$

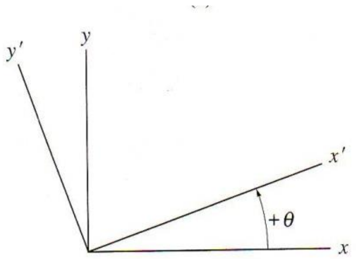
$\theta_{p2} = 28.58$ counterclockwise



Strain at a Point

Plane Strain





$$\varepsilon_{\bar{x}} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_{\bar{y}} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{\bar{x}\bar{y}}}{2} = -\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

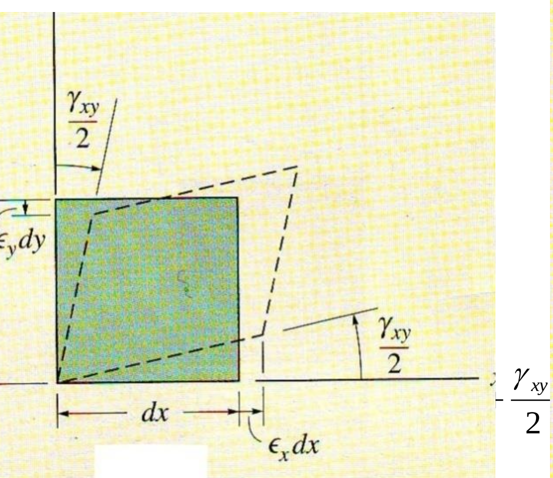
$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$ Principal strain

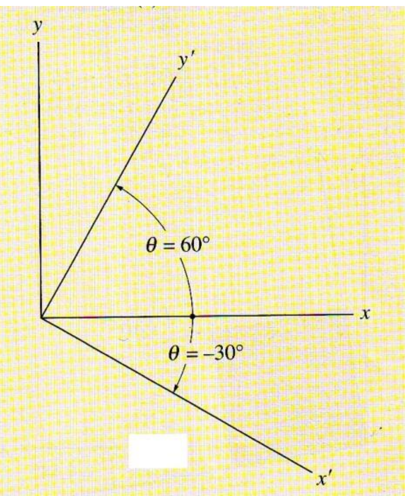
$$\tan 2\theta_s = -\left(\frac{\varepsilon_x - \varepsilon_y}{\gamma_{xy}}\right)$$

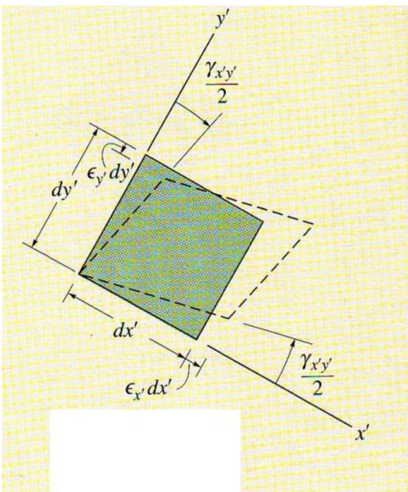
$$\frac{\gamma_{\max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$ Maximum in plane shear strain

$$\varepsilon_{avg} = \frac{\varepsilon_x + \varepsilon_y}{2}$$

Example: A differential element of material at a point is subjected to a state of plane strain $\varepsilon_x = 500\times10^{-6}$, $\varepsilon_y = -300\times10^{-6}$, $\gamma_{xy} = 200\times10^{-6}$, which tends to distort the element as shown below. Determine the equivalent strains acting on an element oriented at the point clockwise 30° from the original position.







$$\varepsilon_{\bar{x}} = \frac{500\times10^{-6} - 300\times10^{-6}}{2} + \frac{500\times10^{-6} + 300\times10^{-6}}{2}\cos(-60) + \frac{200\times10^{-6}}{2}\sin(-60)$$

$$\varepsilon_{\bar{x}} = 213\times10^{-6}$$

$$\varepsilon_{\bar{y}} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_{\bar{y}} = \frac{500\times10^{-6} - 300\times10^{-6}}{2} - \frac{500\times10^{-6} + 300\times10^{-6}}{2}\cos(-60) - \frac{200\times10^{-6}}{2}\sin(-60)$$

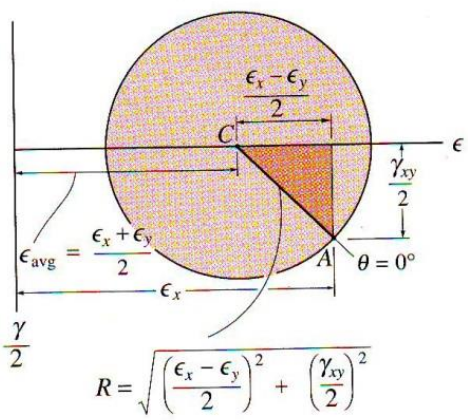
$$\varepsilon_{\bar{y}} = -13.4\times10^{-6}$$

$$\frac{\gamma_{\bar{x}\bar{y}}}{2} = -(\frac{\varepsilon_x - \varepsilon_y}{2})\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$\frac{\gamma_{\bar{x}\bar{y}}}{2} = -(\frac{500\times10^{-6} + 300\times10^{-6}}{2})\sin(-60) + \frac{200\times10^{-6}}{2}\cos(-60)$$

$$\gamma_{\bar{x}\bar{y}} = 793\times10^{-6}$$

Mohr's Circle – Plane Strain

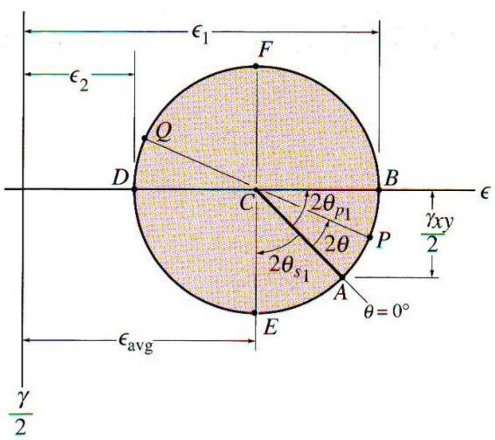


$C = \frac{\varepsilon_x + \varepsilon_y}{2}$

$R = \sqrt{(\frac{\varepsilon_x - \varepsilon_y}{2})^2 + (\frac{\gamma_{xy}}{2})^2}$

Point A ($\varepsilon_x, \frac{\gamma_{xy}}{2}$)

- The principal strain ε_1 and ε_2 are determined from the circle as the coordinates of points B and D.
- The average normal strain and the maximum in plane shear strain are determined from the circle as the coordinates of points E and F.



- The normal and shear strain components $\varepsilon_{\bar{x}}$ and $\gamma_{\bar{x}\bar{y}}$ for a plane specified at angle θ can be obtained from the circle using trigonometry to determine the coordinates of point P.

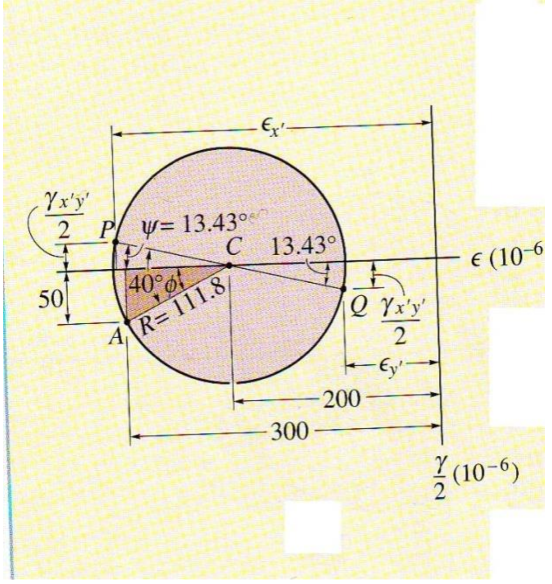
Example:The state of plane strain at a point is represented on an element having components ε_x=-300×10^{-6} ,ε_y=-100×10^{-6}, γ_{xy}=100×10^{-6}. Determine the state of strain on an element oriented 20^{o} clockwise from this reported position.

$$C=\frac{\varepsilon_x+\varepsilon_y}{2}=\frac{-300\times10^{-6}-100\times10^{-6}}{2}=-200\times10^{-6}$$

$$R=\sqrt{(\frac{\varepsilon_x-\varepsilon_y}{2})^2+(\frac{\gamma_{xy}}{2})^2}=\sqrt{(\frac{-300\times10^{-6}+100\times10^{-6}}{2})^2+(\frac{100\times10^{-6}}{2})^2}$$

$$R=111.8\times10^{-6}$$

Point A(-300×10^{-6},50×10^{-6})



$$\phi=\sin^{-1}(\frac{50}{111.8})=26.56$$

$$\psi=40-26.56=13.44$$

$$\varepsilon_{\bar{x}}=-(C+R\cos\psi)$$

$$\varepsilon_{\bar{x}}=-(200\times10^{-6}+111.8\times10^{-6}\cos(13.44))$$

$$\varepsilon_{\bar{x}}=-(309\times10^{-6})$$

$$\frac{\gamma_{\bar{x}\bar{y}}}{2}=-R\sin\psi$$

$$\frac{\gamma_{\bar{x}\bar{y}}}{2}=-111.8\times10^{-6}\sin(13.44)$$

$$\gamma_{\bar{x}\bar{y}}=-52\times10^{-6}$$

$$\varepsilon_{\bar{y}}=-(C-R\cos\psi)$$

$$\varepsilon_{\bar{y}}=-(200\times10^{-6}-111.8\times10^{-6}\cos(13.44))$$

$$\varepsilon_{\bar{y}}=-(91.3\times10^{-6})$$

Theories of Failure:-

1. Ductile Materials

a) Maximum Shear Stress Theory

$$\left.\begin{array}{l}|\sigma_1| = \sigma_Y \\ |\sigma_2| = \sigma_Y\end{array}\right\}$$ σ_1, σ_2 have same signs. ***(Rankine)***

$$\left.|\sigma_1 - \sigma_2| = \sigma_Y\right\}$$ σ_1, σ_2 have opposite signs.***(Guest-Tresca)***

b) Maximum Principal Strain Theory

$$\sigma_1 - \upsilon\sigma_2 - \upsilon\sigma_3 = \sigma_Y$$ ***(Saint-Venant)***

c) Maximum Shear Strain Energy Per Unit Volume (Distortion Energy Theory)

For the case of triaxial stress

$$\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = \sigma_Y^2$$ ***(Maxwell-Huber-Von Mises)***

For the case of plane or biaxial stress

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_Y^2$$

d) Total Strain Energy Per Unit Volume

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\upsilon(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_Y^2$$ ***(Haigh)***

2. Brittle Materials

a) Maximum Normal Stress Theory

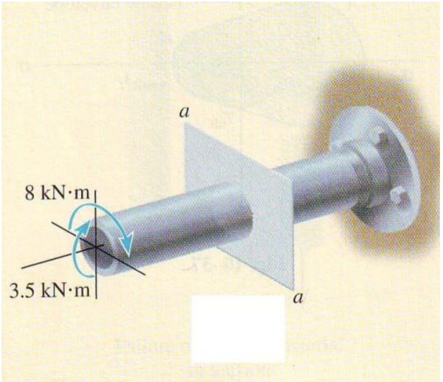
If the material is subjected to plane stress.

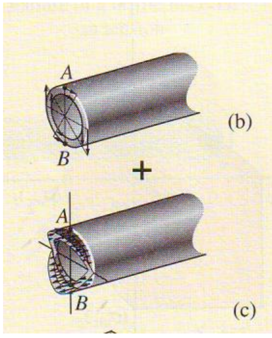
$$|\sigma_1| = \sigma_{ult}$$

$$|\sigma_2| = \sigma_{ult}$$

Example:- The steel pipe shown below has an inner diameter of ***60 mm*** and an outer diameter of ***80 mm***. If it is subjected to a torsional moment of ***8 KN.m*** and a bending moment of ***3.5 KN.m***, determine if these loadings cause failure as defined by the maximum distortion energy theory. The yield stress for the steel found from a tension test is ***σ_Y=250 MPa.***

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 \overset{?}{=} \sigma_Y^2$$

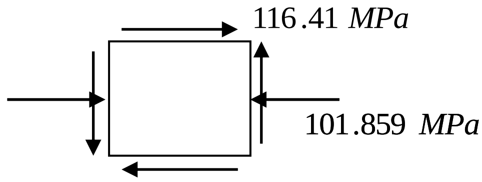




- Point A

$$\tau_A = \frac{Tr}{J} = \frac{8\times 40\times 10^{-3}}{\frac{\pi}{2}[(40\times 10^{-3})^4 - (30\times 10^{-3})^4]} = 116.41 \text{ MPa}$$

$$\sigma_A = \frac{My}{I} = \frac{3.5\times 40\times 10^{-3}}{\frac{\pi}{4}[(40\times 10^{-3})^4 - (30\times 10^{-3})^4]} = 101.859 \text{ MPa}$$

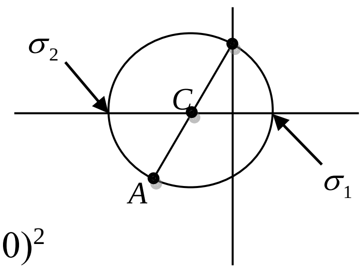


σ_x=-101.859 MPa , σ_y=0 , τ_{xy}=116.41 MPa

c=$\dfrac{\sigma_x+\sigma_y}{2} = \dfrac{-101.859+0}{2} = -50.9295$ MPa

R=$\sqrt{(\dfrac{\sigma_x-\sigma_y}{2})^2+\tau_{xy}^2} = \sqrt{(\dfrac{-101.859-0}{2})^2+(116.41)^2}$ =127.063 MPa

A(-101.859,116.41) Draw Mohr's circle



$\sigma_1 = C + R$=-50.9295+127.063=76.1335 MPa

$\sigma_2 = C - R$=-50.9295-127.063

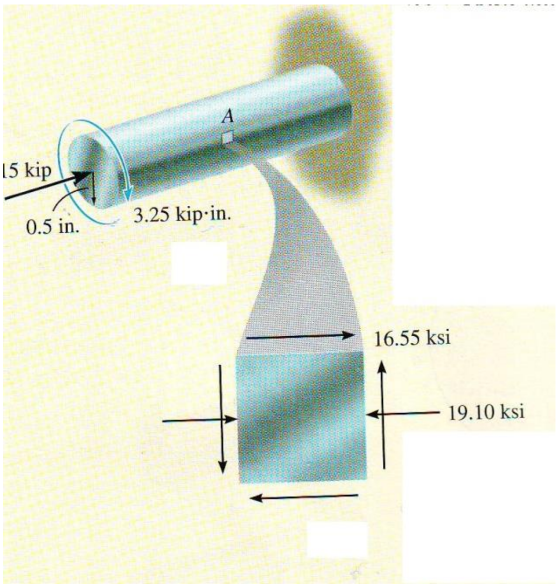
=-177.9925 MPa

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_Y^2$$

$(76.1335)^2-(76.1335)(-177.9925)+(-177.9925)^2 \le (250)^2$

51100≤62500 since 51100<62500 so these loadings will not cause failure.

Example:- The solid shaft shown below has a radius of 0.5 in. and is made of steel having yield stress $\sigma_Y = 36$ ksi. Determine if the loadings cause the shaft to fail according to the maximum shear stress theory and the maximum distortion energy theory.



$$\sigma = \frac{P}{A} = \frac{15}{\pi(0.5)^2} = 19.1\ ksi$$

$$\tau_A = \frac{Tr}{J} = \frac{3.25 \times 0.5}{\frac{\pi}{2}(0.5)^4} = 16.55\ ksi$$

$\sigma_x = -19.1$ ksi , $\sigma_y = 0$, $\tau = 16.55$ ksi

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2}$$

$$= \frac{-19.1+0}{2} \pm \sqrt{(\frac{-19.1-0}{2})^2 + (16.55)^2}$$

$= -9.55 \pm 19.11$

$\sigma_1 = 9.56\ ksi$

$\sigma_2 = -28.66\ ksi$

- Maximum shear stress theory

$|\sigma_1 - \sigma_2| = \sigma_Y$

$|9.56 + 28.66| = 36$

38.2>36

So the failure will occur according to this theory.

- maximum distortion energy theory

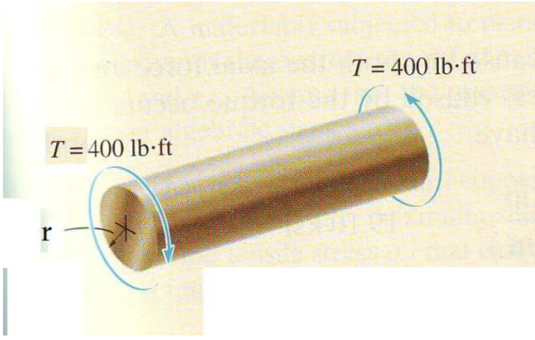
$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_Y^2$

$(9.56)^2-(9.56)(-28.66)+(-28.66)^2=(36)^2$

1186.515<1296

the failure will not occur according to this theory.

Example:- The solid cast iron shaft shown below is subjected to a torque of T=400 lb.ft. Determine the smallest radius so that it does not fail according to the maximum normal stress theory σ_{ult} =20 ksi.



$$\tau = \frac{Tr}{J} = \frac{400 \times 12 \times r}{\frac{\pi}{2}(r)^4} = \frac{3055.8}{r^3}$$

$$\sigma_x = 0 \ , \ \sigma_y = 0 \ , \ \tau = \frac{3055.8}{r^3} \text{ psi}$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2}$$

$$\sigma_{1,2} = \frac{0+0}{2} \pm \sqrt{(\frac{0-0}{2})^2 + (\frac{3055.8}{r^3})^2}$$

$$\sigma_1 = \frac{3055.8}{r^3} \quad , \quad \sigma_2 = -\frac{3055.8}{r^3}$$

$$|\sigma_1| = \sigma_{ult}$$

$$\frac{3055.8}{r^3} = 20000$$

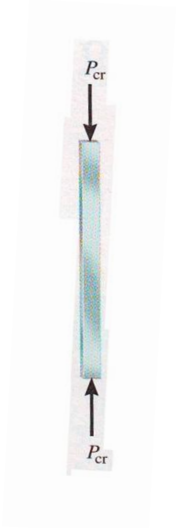
r=0.535 in.

Columns

Columns are long slender members subjected to an axial compressive force. The force may be large enough to cause the member to deflect laterally or sides way, this deflection is called buckling.

Critical Load

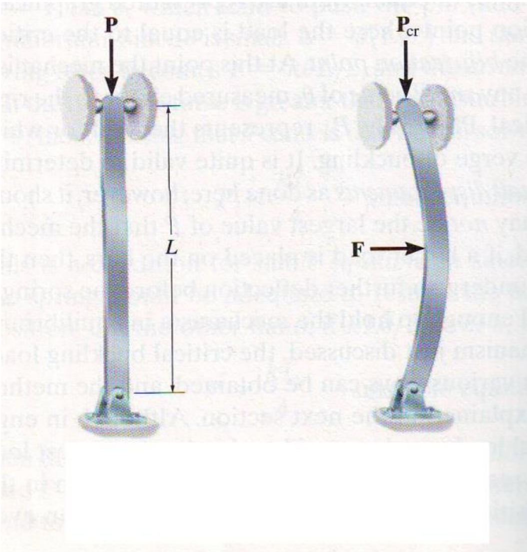
The maximum axial load that a column support when it is on the verge of buckling is called the critical load (P_{cr}).



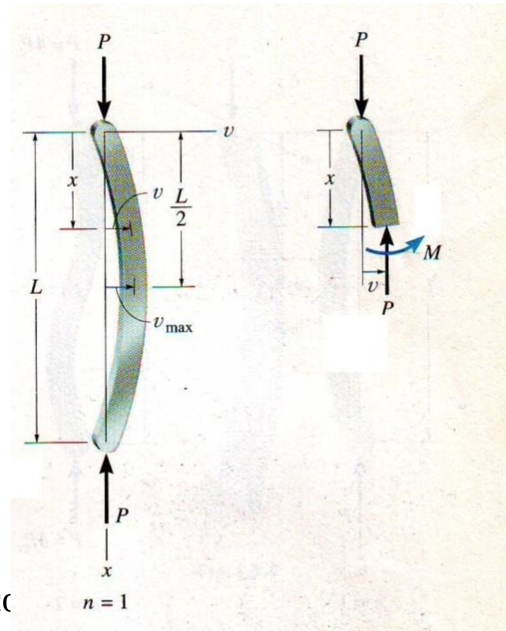
Any additional loading will cause the column to buckle and therefore deflect laterally.

Ideal Column with Pin Supports

The column to beconsideredis an ***ideal column,*** meaning one that is perfectly straight before loading, is made of homogeneous material, and upon which the load is appliedthrough the centroid of the cross section. It is further assumed that the material behaves in a linear-elastic manner and that the column buckles or bends in a single plane.



In order to determine the critical load and the buckled shape of the column we will apply the following equation:



$$EI\frac{d^2v}{dx^2} = M$$

$$\sum M_{section} = 0$$

M+Pv=0

M=-Pv

$$EI\frac{d^2v}{dx^2} = -Pv$$

$$\frac{d^2v}{dx^2} + \left(\frac{P}{EI}\right)v = 0 \qquad \text{.......................(1)}$$

The general solution of equation (1) is:

$$v = C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}x\right) \text{.......................(2)}$$

C_1 and C_2 are determined from the boundary cc

v=0 at x=0 $\Longrightarrow$ $C_2=0$

v=0 at x=L $\Longrightarrow$ $C_1 \sin\left(\sqrt{\frac{P}{EI}}L\right) = 0$

$C_1 \neq 0$ therefore

$$\sin\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

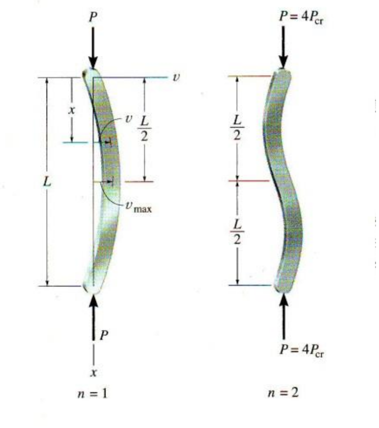
$$\sqrt{\frac{P}{EI}}L = n\pi$$

$$P = \frac{n^2\pi^2 EI}{L^2} \qquad n = 1,2,3,\text{.............}$$

The smallest value of P is obtained when n=1, so the critical load for the column is:

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

This load is sometimes referred to as the Euler load, n represents the number of waves in the deflected shape of the column; if n=2 two waves will appear in the buckled shape and the column will support a critical load that is $4P_{cr}$.

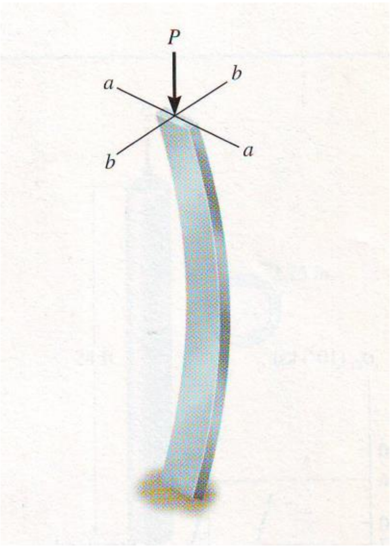


The corresponding buckled shape is:

$$v = C_1 \sin(\frac{\pi x}{L})$$

The constant C_1 represent the maximum deflection v_{max} which occurs at the midpoint of the column.

It is important to realize that the column will buckle about the principal axis of cross section having the least moment of inertia(the weakest axis). For example a column having a rectangular cross section as shown below will buckle about the a-a axis not the b-b axis.



As a result engineers usually try to achieve a balance keeping the moments of inertia the same in all directions

$$I_x \approx I_y$$

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

P_{cr}: critical or maximum axial load on the column just before it begins to buckle. This load must not cause the stress in the column to exceed the proportional limit.
E: modulus of elasticity for the material.
I: least moment of inertia for the column's cross sectional area.
L: unsupported length of the column, whose ends are pinned.
$I=Ar^2$

$$\sigma_{cr}=\frac{\pi^2 E}{(\frac{L}{r})^2}$$

σ_{cr} :critical stress which is an average stress in the column just before the column buckles. This stress is an elastic stress and therefore:

$$\sigma_{cr}\le\sigma_Y$$

r: smallest radius of gyration of the column $r=\sqrt{\frac{I}{A}}$.
L/r: slenderness ratio, it's a measure of the column flexibility.

Example: A 24 ft long A-36 steel tube having the cross section shown below is to be used as a pin ended column. Determine the maximum allowable axial load the column can support so that it does not buckle. $E_{st}=29\times10^3$ksi, σ_Y =36 ksi.

$$P_{cr}=\frac{\pi^2 EI}{L^2}=\frac{\pi^2\times29\times10^3\times\frac{\pi}{4}(3^4-2.75^4)}{(24\times12)^2}$$

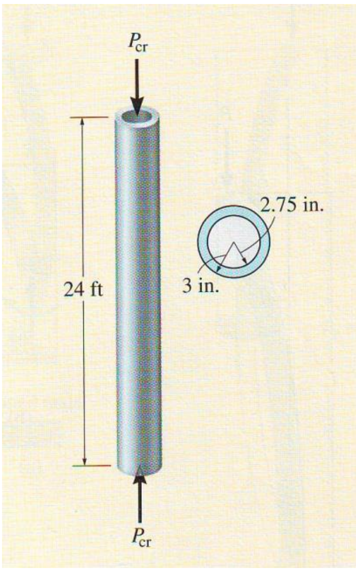
=64.52 kip.

$$\sigma_{cr}=\frac{P_{cr}}{A}=\frac{64.52}{\pi(3^2-2.75^2)}$$

=14.28 ksi

Since $\sigma_{cr}<\sigma_Y$

P_{allow}=64.52 kip.



Example: The A-36 steel W8×31 member shown below is to be used as a pin connected column. Determine the largest axial load it can support before it either begins to buckle or the steel yields. $E_{st}=29\times10^3$ksi, σ_Y =36 ksi. A=9.13 in^2, I_x=110 in^4, I_y=37.1 in^4.

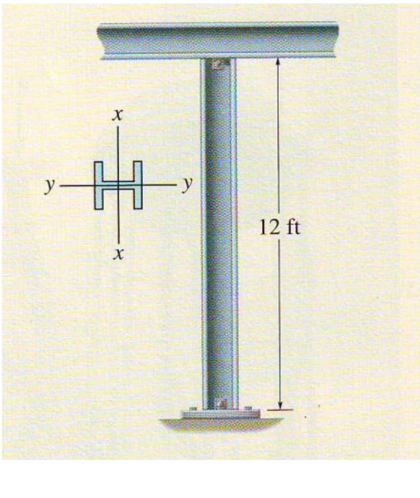
Buckling occurs about y-axis.

$$P_{cr}=\frac{\pi^2 EI}{L^2}=\frac{\pi^2\times29\times10^3\times37.1}{(12\times12)^2}$$

=512 kip

$$\sigma_{cr}=\frac{P_{cr}}{A}=\frac{512}{9.13}=56\ ksi$$

$$\sigma_{cr}>\sigma_Y$$

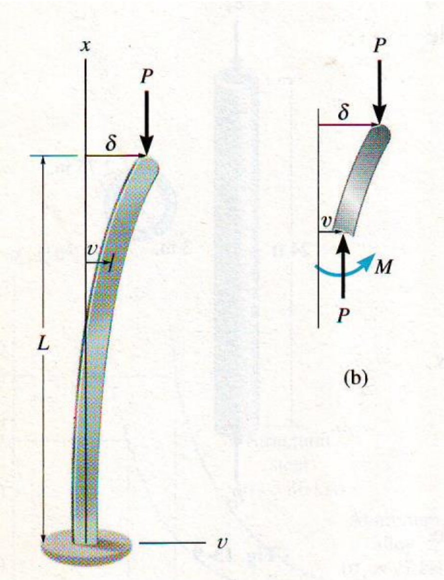


$$\sigma_y = 36 = \frac{P}{A} = \frac{P}{9.13}$$

P=328.68 kip.

Columns Having Various Types of Supports

- Fixed-Free column



$$\sum M_{section} = 0$$

M-P(δ -v)=0

M=P(δ -v)

$$EI\frac{d^2v}{dx^2} = \text{P}(\delta - \text{v})$$

$$\frac{d^2v}{dx^2} + \frac{P}{EI}v = \frac{\text{P}}{EI}\delta \quad \text{......................(1)}$$

The solution of equation (1) consists of both a complemen

$$v = C_1 \sin(\sqrt{\frac{P}{EI}}x) + C_2 \cos(\sqrt{\frac{P}{EI}}x) + \delta \quad \text{......................(2)}$$

C_1 and C_2 are determined from the boundary conditions :

v=0 at x=0 $\Rightarrow$ $C_2=-\delta$

$\frac{dv}{dx}$=0 at x=0

$$\frac{dv}{dx} = C_1\sqrt{\frac{P}{EI}}\cos(\sqrt{\frac{P}{EI}}x) - C_2\sqrt{\frac{P}{EI}}\sin(\sqrt{\frac{P}{EI}}x)$$

C_1=0

$$v = \delta[1 - \cos(\sqrt{\frac{P}{EI}}x)] \quad \text{.................(3)}$$

Since the deflection at the top of the column is δ , that is at x=L v= δ

$$\delta \cos(\sqrt{\frac{P}{EI}}L) = 0$$

$\delta \neq 0$

$$\cos(\sqrt{\frac{P}{EI}}L) = 0 \quad \text{or} \quad \sqrt{\frac{P}{EI}}L = \frac{n\pi}{2}$$

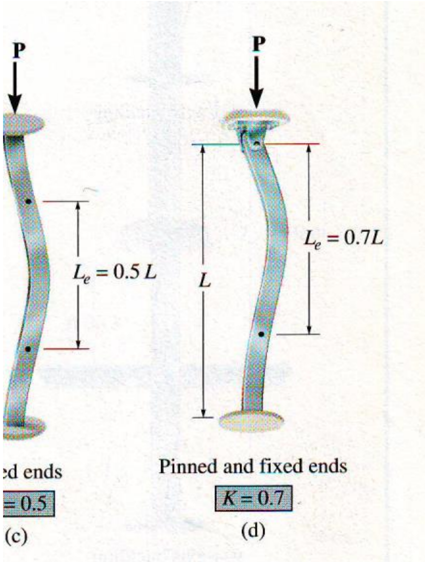
$$P = \frac{n^2\pi^2 EI}{4L^2}$$

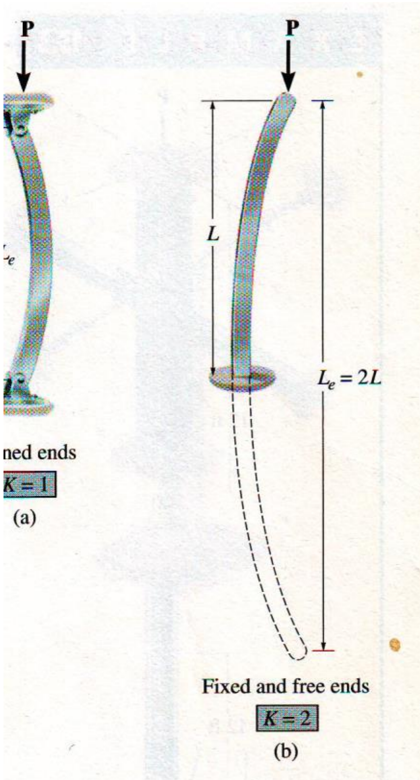
The smallest value of P is obtained when n=1, so the critical load for the column is:

$$P_{cr} = \frac{\pi^2 EI}{4L^2}$$

Effective Length

The effective length (L_e) is the distance between points of inflection (that is , points of zero moment) in its deflection curve, assuming that the curve is extended (if necessary) until points of inflection are reached.





L_e=KL

Pinned –Ends K=1

Fixed-Free Ends K=2

Fixed-Ends K=0.5

Pinned-Fixed Ends K=0.7

Euler's formula becomes:

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} \qquad ; \sigma_{cr} = \frac{\pi^2 E}{(\frac{KL}{r})^2}$$

KL/r: columns effective slenderness ratio.

For fixed-Free ends K=2

$$P_{cr} = \frac{\pi^2 EI}{4L^2}$$

Example: A W6×15 steel column is 24 ft long and is fixed at is ends as shown below. Its load carrying capacity is increased by bracing it about the y-y (weak) axis using strut that are assumed to be pin connected to its midheight. Determine the load it can support so that the column does not buckle nor the material exceed the yield stress. Take $E_{st}=29\times10^3$ksi and σ_Y =60 ksi. A=4.43 in^2, I_x=29.1 in^4, I_y=9.32 in^4.

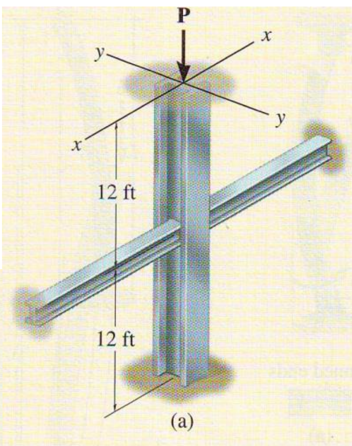
$$(P_{cr})_x = \frac{\pi^2 EI_x}{(KL)_x^2} = \frac{\pi^2 \times 29\times10^3 \times 29.1}{(12\times12)^2} = 401.7\ kip$$

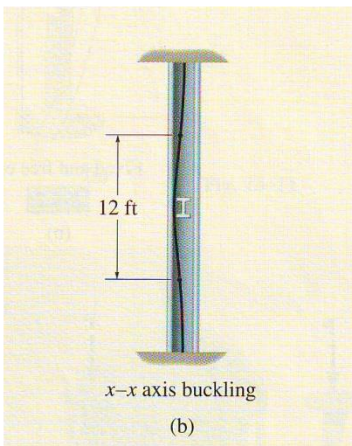
$$(P_{cr})_y = \frac{\pi^2 EI_y}{(KL)_y^2} = \frac{\pi^2 \times 29\times10^3 \times 9.32}{(0.7\times12\times12)^2} = 262.5\ kip$$

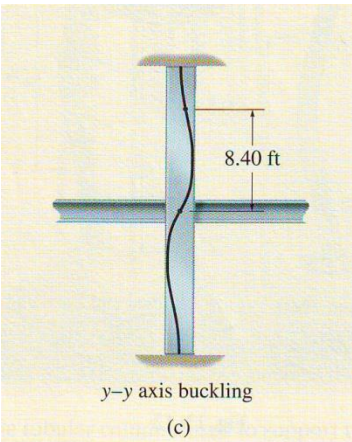
$$\sigma_{cr} = \frac{(P_{cr})_y}{A} = \frac{262.5}{4.43} = 59.3\ ksi$$

$$\sigma_{cr} < \sigma_Y$$

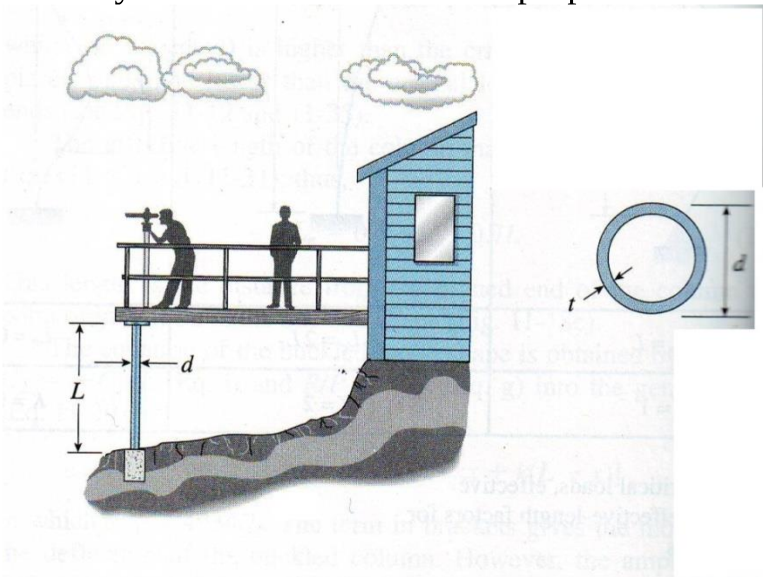
P_{cr}=262.5 kip.







Example: A viewing platform in a wild animal park is supported by a raw of aluminum pipe columns having length 3.25 m and outer diameter 100 mm. The bases of the columns are set in concrete footings and the tops of the columns are supported laterally by the platform (pinned). The columns are being designed to support compressive loads 100 kN. Determine the minimum required thickness t of the columns if a factor of safety n=3 is required with respect to Euler buckling for aluminum use 72 GPa for the modulus of elasticity and use 480 MPa for the proportional limit.



For fixed –pinned ends column

$$P_{cr}=\frac{\pi^2 EI}{(0.7L)^2}$$

$P_{cr}=nP=3\times100=300$ kN

$$300=\frac{\pi^2\times72\times10^6\times I}{(0.7\times3.25)^2}$$

$I=2.185\times10^{-6}\ m^4$

$$I=\frac{\pi}{64}(d_o^4-d_i^4)$$

$$2.185\times10^{-6}=\frac{\pi}{64}[(100\times10^{-3})^4-(100\times10^{-3}-2t)^4]$$

$t=6.846\times10^{-3}$ m

$t=6.846$ mm

$d_i=86.308$ mm

$$A=\frac{\pi}{4}(d_o^2-d_i^2)=\frac{\pi}{4}[(100\times10^{-3})^2-(86.308\times10^{-3})^2]$$

$A=2.0034\times10^{-3}\ m^2$

$$\sigma_{cr}=\frac{P_{cr}}{A}=\frac{300}{2.0034\times10^{-3}}=149.738\ MPa$$

$$\sigma_{cr}<\sigma_Y$$

$t=6.846$ mm