***Strength of materials*** is a branch of applied mechanics that deals with the behavior of solid bodies subjected to various types of loading. The solid bodies include axially-loaded bars, shafts, beams, and columns. The objective of analysis will be the determination of the stresses, strains, and deformations produced by the loads.

## Simple Stress (σ):

If a cylindrical bar is subjected to a direct ***pull*** or ***push*** along its axis, then it is said to be subjected to ***tension*** or ***compression***.


Tension Compression

In SI systems of units load is measured in ***Newton (N)*** or ***KiloNewton (KN)*** or ***Meganewton (MN).***

***Normal stress (σ) :*** is the intensity of normal force per unit area

$$\text{Stress} = \frac{Load}{Area}$$

$$\sigma = \frac{P}{A}$$

stress may thus be ***compressive*** or ***tensile*** depending on the nature of the load and will be measured in units of Newton per square meter ***(N/m²)***. This unit, called ***Pascal***

1 Pa=1 $N/m^2$
1 KPa=1000 Pa=$10^3$ Pa
1 MPa=$10^6$ Pa
1 GPa=$10^9$ Pa

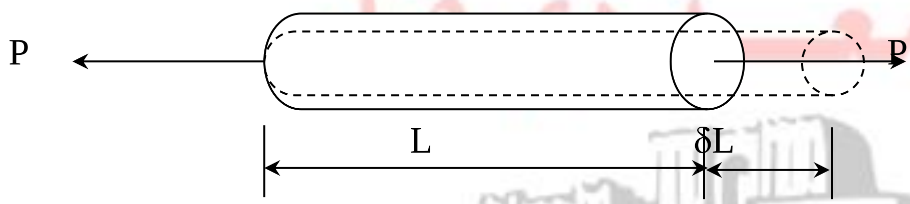
In the U.S. customary or foot-pound-second system of units, express stress in pounds per square inch ***(Psi)*** or kilopound per square inch ***(Ksi)***

## Normal Strain (ε):

If a bar is subjected to a direct load, and hence a stress, the bar will change in length. If the bar has an original length (L) and changes in length by an amount (δL), the strain produced is defined as follows:

$$\text{Strain}(\varepsilon)=\frac{change\,in\,length}{original\,length}$$

$$\varepsilon=\frac{\delta L}{L}$$



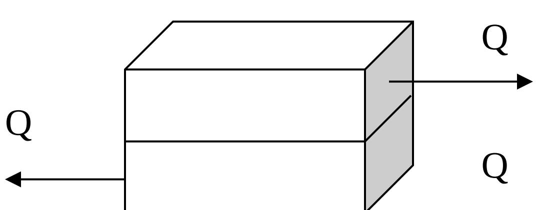

Strain is thus a measure of the deformation of the material and is non-dimensional, i.e. it has no units. Tensile stresses and strains are considered positive sense. Compressive stresses and strains are considered negative in sense.

## Shear Stress (τ) and Bearing Stress ($\sigma_b$):

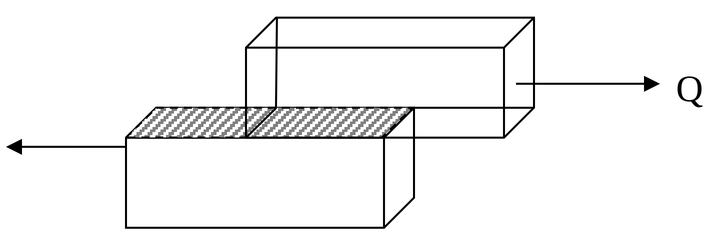
Shearing stress differs from both tensile and compressive stress in that it is caused by forces acting along or parallel to the area resisting the forces, whereas tensile and compressive stresses are caused by forces perpendicular to the areas on which they act. For this reason, tensile and compressive stresses are called normal stresses, whereas a shearing stress may be called a tangential stress.

A shearing stress is produced whenever the applied loads cause one section of a body to tend to slide past its adjacent section.

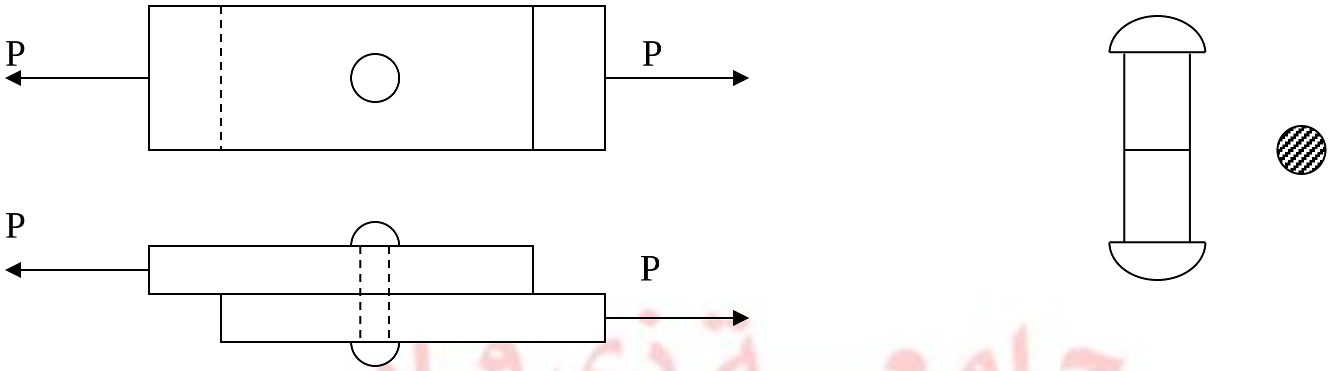
$$\text{Shear stress}=\frac{Shear\,load}{Area\,resisting\,shear}$$




$$\tau=\frac{Q}{A}$$

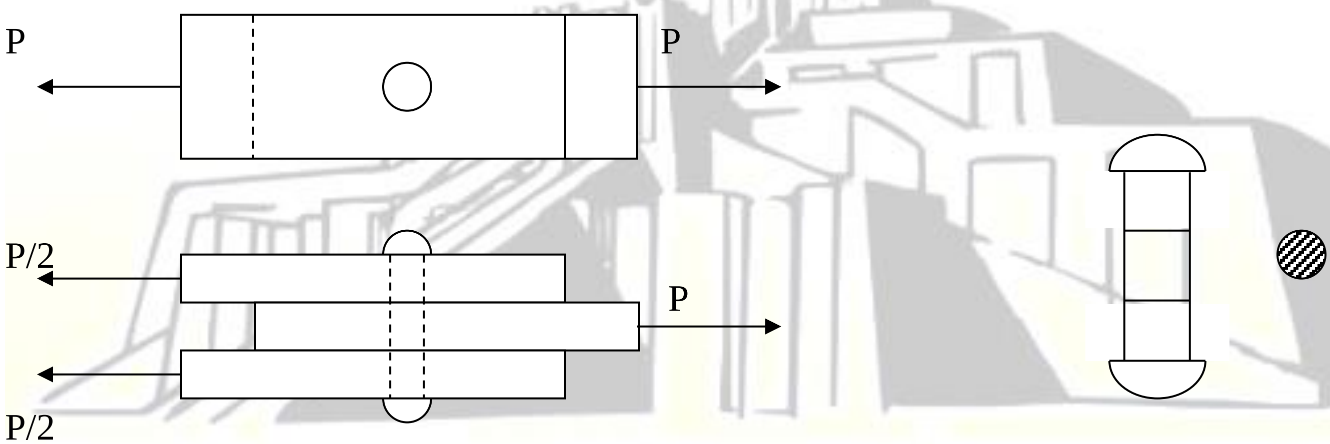


Area resisting shear is the shaded area as shown above.




Single shear stress

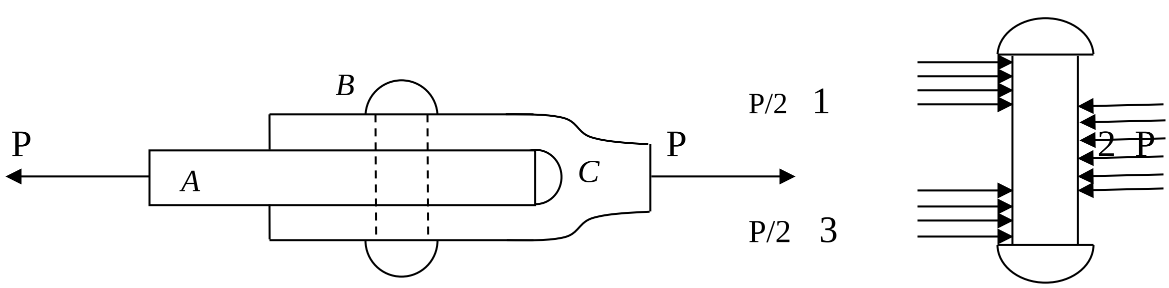
$$\tau = \frac{P}{A}$$




Double shear stress

$$\tau = \frac{P/2}{A}$$

Bearing stress is a normal stress that is produced by the compression of one surface against another. The bearing area is defined as the projected area of the curved bearing surface.

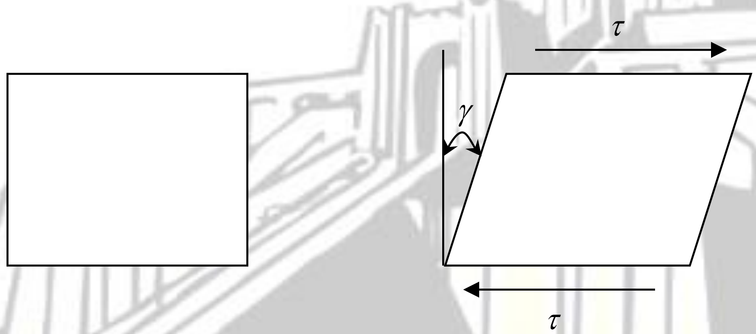


$$\sigma_b = \frac{F_b}{A_b}$$

Consider the bolted connection shown above, this connection consists of a flat bar A, a clevis C, and a bolt B that passes through holes in the bar and clevis. Consider the bearing stresses labeled 1, the projected area $A_b$ on which they act is rectangle having a height equal to the thickness of the clevis and a width equal to the diameter of the bolt, the bearing force $F_b$ represented by the stresses labeled 1 is equal to P/2. The same area and the same force apply to the stresses labeled 3. For the bearing stresses labeled 2, the bearing area $A_b$ is a rectangle with height equal to the thickness of the flat bar and width equal to the bolt diameter. The corresponding bearing force $F_b$ is equal to the load P.

## Shear Strain ( γ ):

Shear strain is a measure of the distortion of the element due to shear. Shear strain is measured in radians and hence is non-dimensional, i.e. it has no units.




## Elastic Materials-Hook's Law:

A material is said to be elastic if it returns to its original, when load is removed. In elastic material, stress is proportional to strain. Hook's law therefore states that:

Stress ($\sigma$ ) $\propto$ strain ( $\varepsilon$ )

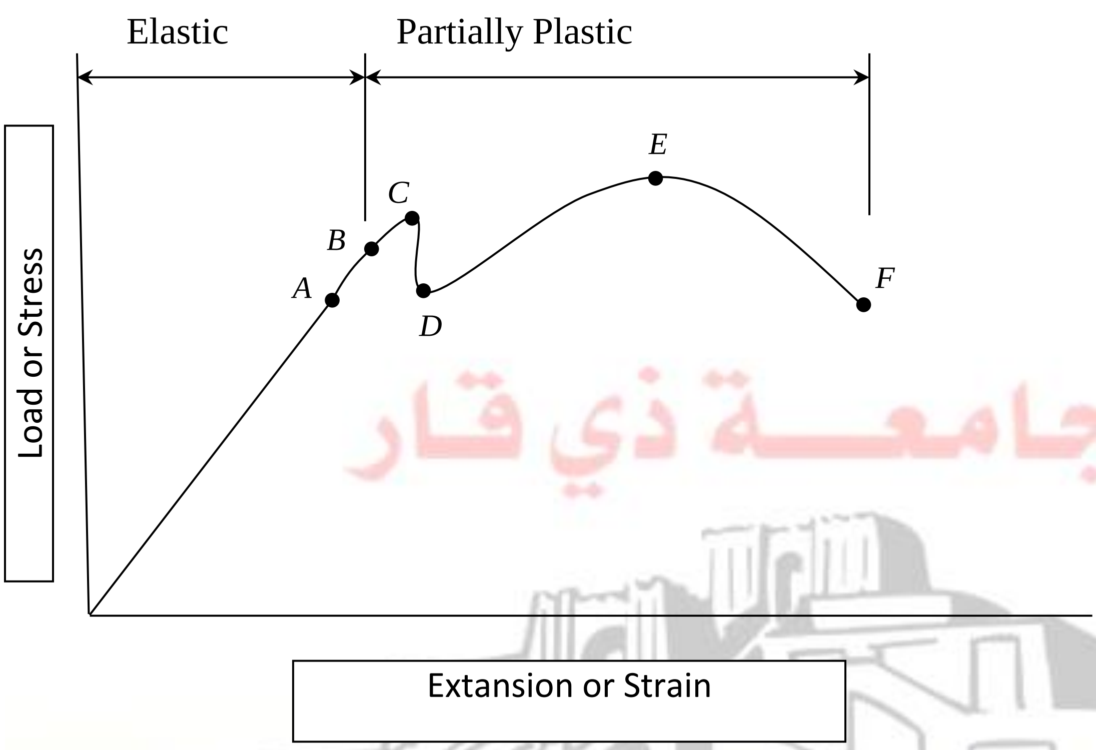
$$\frac{stress}{strain} = \text{constant}$$

Within the elastic limit, i.e. within the limits in which Hook's law applies, it has been shown that:

$$\frac{\sigma}{\varepsilon} = E$$

This constant is given the symbol E and termed the modulus of elasticity or Young's modulus.

## The Stress-Strain Diagram:




Point A is termed to limit proportionality. Point B is termed to elastic limit. Beyond the elastic limit plastic deformation occurs and strains are not totally recoverable. The point C termed the upper yield point and D the lower yield point, relatively rapid increase in strain occur without correspondingly high increase in load or stress.

Beyond the yield point some increase in load is required to take the strain to point E on the graph. Between D and E the material is said to be in elastic-plastic state, some of the section remaining elastic and hence contributing to recovery of the original dimensions if load is removed, the reminder being plastic. Beyond E the cross sectional area of the bar beings to reduce rapidly over a relatively small length of the bar and the bar is said to neck. This necking takes place whilst the load reduces and fracture of the bar finally occurs at point F.

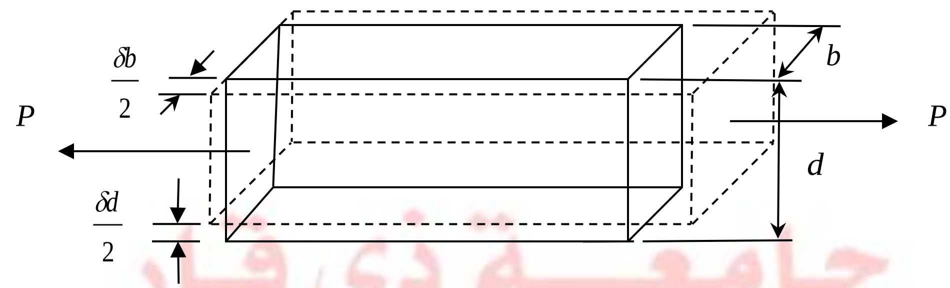
The nominal stress at failure, termed the maximum or ultimate tensile stress, is given by the load at E divided by the original cross-sectional area of the bar.

## Poisson's Ratio ($\upsilon$ ):

Consider the rectangular bar shown below subjected to a tensile load. Under the action of this load the bar will increase in length by an amount $\delta L$ giving a longitudinal strain in the bar of:

$$\varepsilon_L = \frac{\delta L}{L}$$

The bar will also exhibit a reduction in dimensions laterally i.e. its breadth and depth will both reduce.




The associated lateral strains will both be equal, will be of opposite sense to the longitudinal strain, and will be given by:

$$\varepsilon_{lat} = -\frac{\delta d}{d} = -\frac{\delta b}{b}$$

Poisson's ratio is the ratio of the lateral and longitudinal strains and always constant

Poisson's ratio= $\frac{Lateral\,Strain}{Longitudinal\,Strain}$

$$\nu = \frac{\delta d / d}{\delta L / L}$$

Longitudinal Strain= $\frac{\sigma}{E}$

Lateral Strain= $\upsilon\frac{\sigma}{E}$

## Modulus of Rigidity ( G ):

For materials within the elastic range the shear strain is proportional to the shear stress producing it.

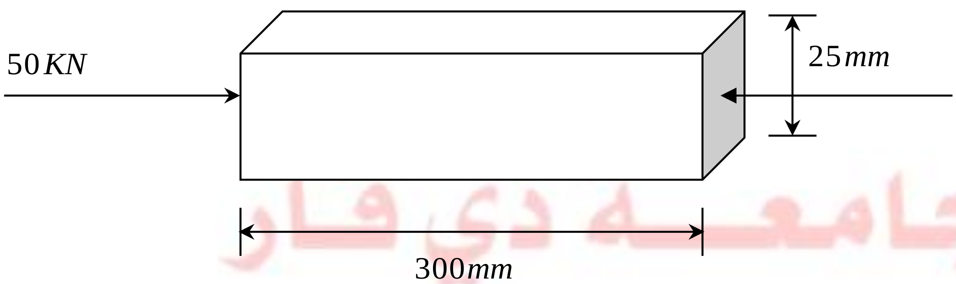
$$\tau \propto \gamma$$

$\frac{Shear\,Stress}{Shear\,Strain}$ =Constant

$\frac{\tau}{\gamma}$ =G

The constant G is termed the modulus of rigidity.

***Example 1:***A ***25 mm*** square cross-section bar of length ***300 mm*** carries an axial compressive load of ***50 KN***. Determine the stress set up in the bar and its change of length when the load is applied. For the bar material ***E=200 GN/m²***.




Cross-section area of the bar(A)=$25\times10^{-3}\times25\times10^{-3}=625\times10^{-6}$ $m^2$

$$\sigma = \frac{P}{A}$$

$$\sigma=\frac{50\times10^{3}}{625\times10^{-6}}=80000000 \text{ N/m}^2$$

$\sigma$=80 MN/ $m^2$

$$\varepsilon = \frac{\sigma}{E}$$

$$\varepsilon=\frac{80\times10^{6}}{200\times10^{9}}=0.0004$$

$$\delta L = \varepsilon L$$

$\delta L=0.0004\times300\times10^{-3}=0.12\times10^{-3}$m

$\delta L$=0.12 mm

***Example 2:*** Two circular bars, one of brass and the other of steel, are to be loaded by a shear load of ***30 KN***. Determine the necessary diameter of the bars a) in single shear b) in double shear, if the shear stress in the two materials must not exceed ***50 MN/m²*** and ***100 MN/m²*** respectively.

***a) Single Shear***

$$\tau = \frac{F}{A} \Longrightarrow A = \frac{F}{\tau}$$

- ***For brass material***

$$A = \frac{30\times10^{3}}{50\times10^{6}} = 0.0006\ m^2$$

$$A = \pi r^2 \Longrightarrow r = \sqrt{\frac{A}{\pi}} \Longrightarrow r = \sqrt{\frac{0.0006}{\pi}}$$

$r = 13.8197\times10^{-3}$ m

the diameter of the bar (d)=$27.639\times10^{-3}$ m

- ***For steel material***

$$A = \frac{30\times10^{3}}{100\times10^{6}} = 0.0003\ m^2$$

$$r = \sqrt{\frac{0.0003}{\pi}} = 9.772\times10^{-3}\ m$$

the diameter of the bar (d)=$19.544\times10^{-3}$ m

***b) Double Shear***

$$\tau = \frac{F}{2A} \Longrightarrow A = \frac{F}{2\tau}$$

- ***For brass material***

$$A = \frac{30\times10^{3}}{2\times50\times10^{6}} = 0.0003\ m^2$$

$$r = \sqrt{\frac{0.0003}{\pi}} = 9.772\times10^{-3}\ m$$

the diameter of the bar (d)=$19.544\times10^{-3}$ m

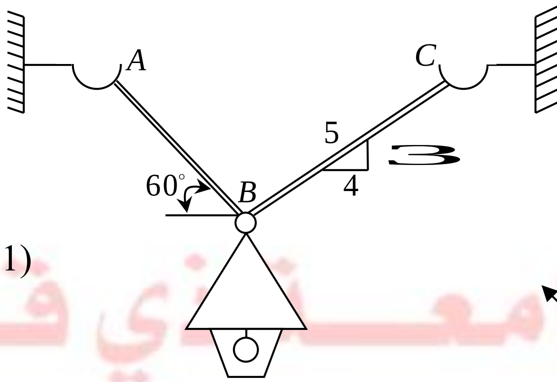
- ***For steel material***

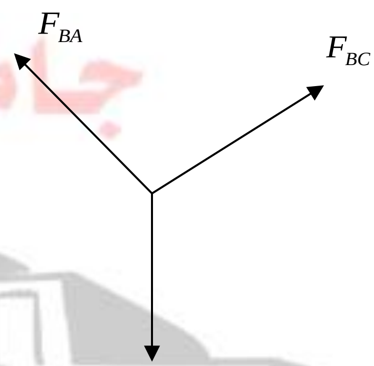
$$A = \frac{30\times10^{3}}{2\times100\times10^{6}} = 0.00015\ m^2$$

$$r = \sqrt{\frac{0.00015}{\pi}} = 6.909\times10^{-3}\ m$$

the diameter of the bar (d)=$13.819\times10^{-3}$ m

***Example 3:*** The ***80 kg*** lamp is supported by two rods ***AB*** and ***BC*** as shown. If ***AB*** has a diameter of ***10 mm*** and ***BC*** has a diameter of ***8 mm***, determine the average normal stress in each rod.




$$\sum F_x = 0$$

$$F_{BC}\times\frac{4}{5}-F_{BA}\times\cos 60=0$$

$F_{BC}=0.625F_{BA}$ ……………..(1)

$$\sum F_y = 0$$

$$F_{BC}\times\frac{3}{5}+F_{BA}\times\sin 60-784.8=0$$

$F_{BC}=1308-1.44337F_{BA}$ ……….(2)

$1308-1.44337F_{BA}=0.625F_{BA}$

$F_{BA}=632.38$ N

$F_{BC}=395.2375$ N

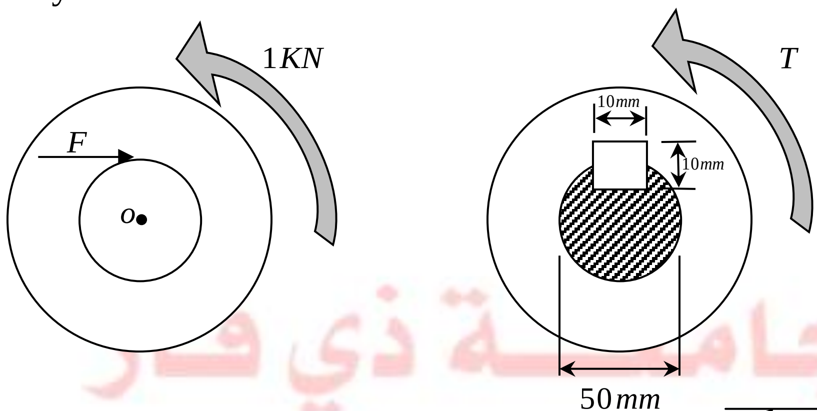
$$\sigma_{BA}=\frac{F_{BA}}{A_{BA}}=\frac{632.38}{\pi(5\times10^{-3})^2}$$

$\sigma_{BA}=8.051877\times10^{6}$ Pa
$\sigma_{BA}=8.051877$ MPa

$$\sigma_{BC}=\frac{F_{BC}}{A_{BC}}=\frac{395.2375}{\pi(4\times10^{-3})^2}$$

$\sigma_{BC}=7.863149\times10^{6}$ Pa
$\sigma_{BC}=7.863149$ MPa

***Example 4:*** Shafts and pulleys are usually fastened together by means of a key, as shown. Consider a pulley subjected to a turning moment ***T*** of ***1 KN.m*** keyed by a ***10 mm×10 mm×75 mm*** key to the shaft. The shaft is ***50 mm*** in diameter. Determine the shear stress on a horizontal plane through the key.



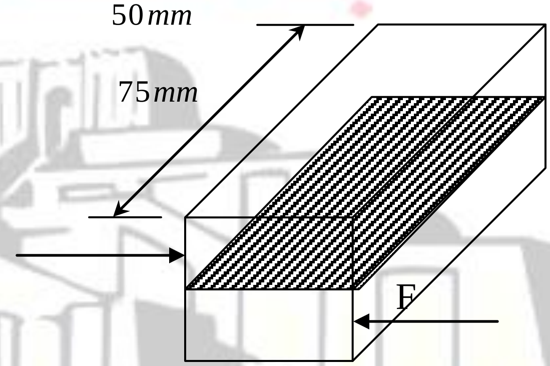

$\sum M_o = 0$

$1\times10^3 - F\times0.025 = 0$

F=40000 N

F=40 KN

$\tau = \frac{F}{A}$




A is the shaded area

$\tau = \frac{40\times10^3}{10\times10^{-3}\times75\times10^{-3}}$

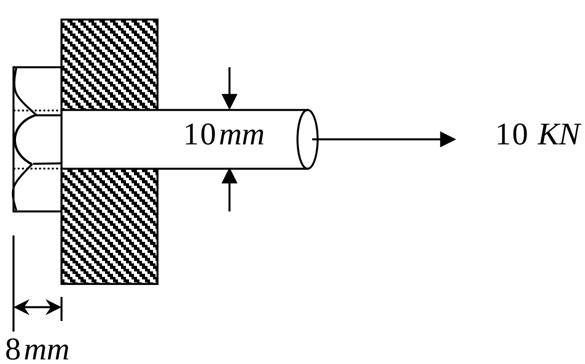
$\tau = 53.333\times10^6$ N/m$^2$

$\tau = 53.333$ MN/m$^2$

***Example 5:*** Consider a steel bolt ***10 mm*** in diameter and subjected to an axial tensile load of ***10 KN*** as shown. Determine the average shearing stress in the bolt head, assuming shearing on a cylindrical surface of the same diameter as the bolt.

A=πdt

$A = \pi\times10\times10^{-3}\times8\times10^{-3} = 0.000251327$ m$^2$




$\tau = \frac{F}{A}$

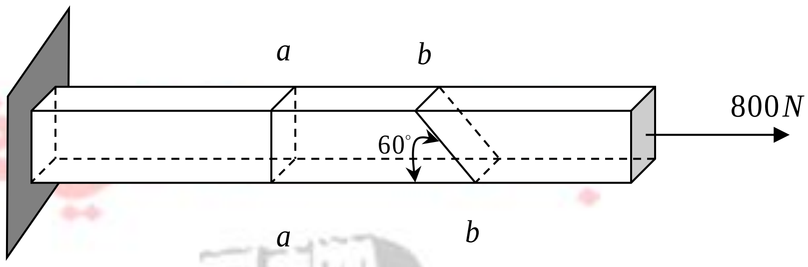
$\tau = \frac{10\times10^3}{0.000251327}$

$\tau = 39.7888\times10^6$ N/m$^2$

$\tau = 39.7888$ MN/m$^2$

***Example 6:*** The bar shown has a square cross section for which the depth and thickness are ***40 mm***. If an axial force of ***800 N*** is applied along the centroidal axis of the bar's cross sectional area, determine the average normal stress and average shear stress acting on the material along a) section plane ***a-a*** and b) section plane ***b-b***.

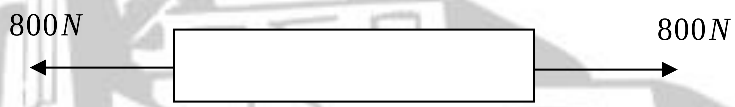
a) section plane ***a-a***




$$\sigma = \frac{P}{A}$$

$$\sigma = \frac{800}{40\times10^{-3}\times40\times10^{-3}}$$

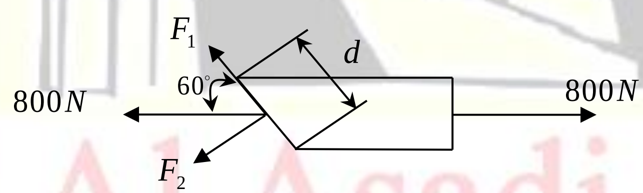
$\sigma$=500 KN/m$^2$




$$\tau = \frac{F}{A}$$

F=0
$\tau$=0

b) section plane ***b-b***




$$d = \frac{40}{\sin 60} = 46.188 \text{ mm}$$

$$\sigma = \frac{F_2}{A}$$

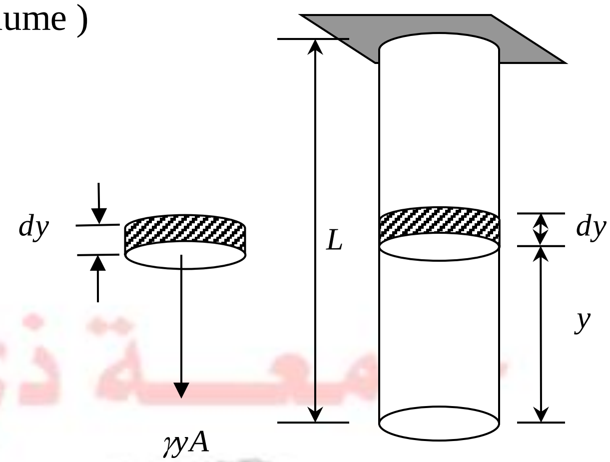
$$\sigma = \frac{800\sin 60}{46.188\times10^{-3}\times40\times10^{-3}} = 375 \text{ KN/m}^2$$

$$\tau = \frac{F_1}{A}$$

$$\tau = \frac{800\cos 60}{46.188\times10^{-3}\times40\times10^{-3}} = 216.50645 \text{ KN/m}^2$$

***Example 7:*** Determine the total increase of length of a bar of constant cross section hanging vertically and subject to its own weight as the only load. The bar is initially straight.

γ: is the specific weight ( weight/unit volume )
A: is the cross-sectional area




$$d\delta = \frac{\gamma A y dy}{AE}$$

$$\delta = \int_0^L d\delta$$

$$\delta = \int_0^L \frac{\gamma A y dy}{AE}$$

$$= \frac{\gamma A}{AE} \int_0^L y dy$$

$$= \frac{\gamma A}{AE} \frac{1}{2} y^2 \Big|_0^L$$

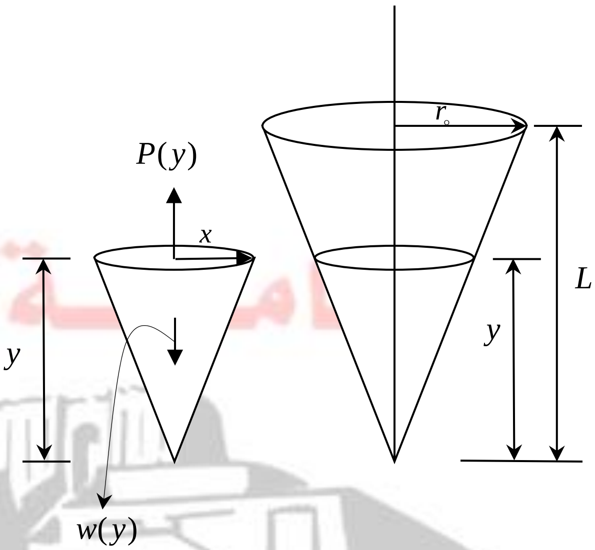
$$\delta = \frac{\gamma A}{2AE} L^2$$

$$\delta = \frac{\gamma AL.L}{2AE}$$

W=γAL

$$\delta = \frac{W.L}{2AE}$$

***Example 8:*** A member is made from a material that has a specific weight $\gamma$ and modulus of elasticity ***E***. If its formed into a cone having the dimensions shown, determine how far its end is displaced due to gravity when its suspended in the vertical position.




$$\frac{x}{r_\circ} = \frac{y}{L}$$

$$x = r_\circ \frac{y}{L}$$

$$v = \frac{\pi}{3} x^2 y$$

$$w(y) = \gamma v = \gamma \frac{\pi}{3} x^2 y$$

$$= \frac{\pi}{3} \gamma \frac{r_\circ^2 y^2}{L^2} y$$

$$w(y) = \frac{\pi}{3} \frac{\gamma r_\circ^2}{L^2} y^3$$

From equilibrium P(y)=w(y)

$$P(y) = \frac{\pi}{3} \frac{\gamma r_\circ^2}{L^2} y^3$$

$$A(y) = \pi x^2 = \pi \frac{r_\circ^2}{L^2} y^2$$

$$d\delta = \frac{P(y)dy}{A(y)E} = \frac{\frac{\pi}{3} \gamma \frac{r_\circ^2}{L^2} y^3 dy}{\pi \frac{r_\circ^2}{L^2} y^2 E}$$

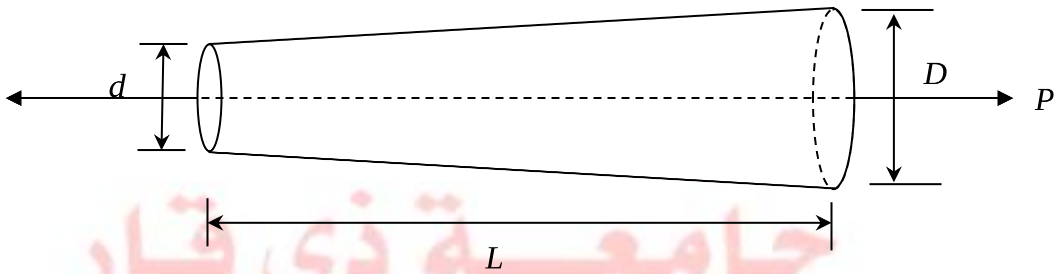
$$d\delta = \frac{\gamma}{3E} y dy$$

$$\delta = \int_0^L d\delta = \int_0^L \frac{\gamma y dy}{3E}$$

$$\delta = \frac{\gamma L^2}{6E}$$

***Example 9:*** A solid truncated conical bar of circular cross section tapers uniformly from a diameter d at its small end to D at the large end. The length of the bar is L. Determine the elongation due to an axial force P applied at each end as shown.

$$r=\frac{d}{2}+y$$

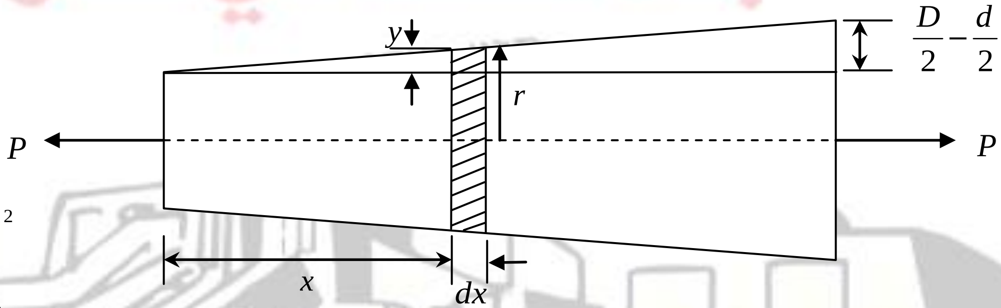



$$\frac{y}{x}=\frac{\frac{D}{2}-\frac{d}{2}}{L}$$

$$y=(\frac{D}{2}-\frac{d}{2})\frac{x}{L}$$

$$r=\frac{d}{2}+(\frac{D}{2}-\frac{d}{2})\frac{x}{L}$$

$$A(x)=\pi r^2$$




$$A(x)=\pi\left[\frac{d}{2}+(\frac{D}{2}-\frac{d}{2})\frac{x}{L}\right]^2$$

$$d\delta=\frac{Pdx}{A(x)E}=\frac{Pdx}{\pi\left[\frac{d}{2}+(\frac{D}{2}-\frac{d}{2})\frac{x}{L}\right]^2 E}$$

$$\delta=\int_0^L\frac{Pdx}{\pi\left[\frac{d}{2}+(\frac{D}{2}-\frac{d}{2})\frac{x}{L}\right]^2 E}$$

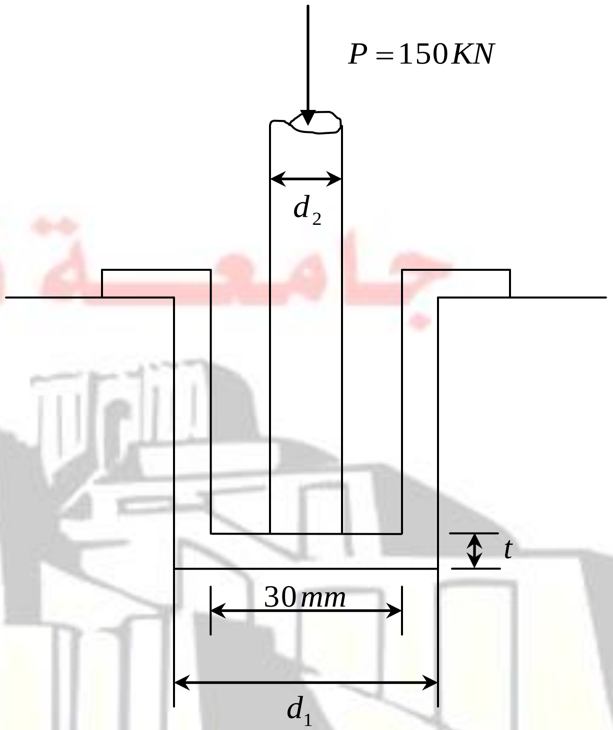
$$=-\frac{P}{E\frac{\pi}{L}(\frac{D}{2}-\frac{d}{2})}\left[\frac{d}{2}+(\frac{D}{2}-\frac{d}{2})\frac{x}{L}\right]^{-1}\Bigg|_0^L$$

$$=-\frac{PL}{E\pi(\frac{D}{2}-\frac{d}{2})\left[\frac{d}{2}+(\frac{D}{2}-\frac{d}{2})\frac{x}{L}\right]}\Bigg|_0^L=-\frac{PL}{E\pi(\frac{D}{2}-\frac{d}{2})\left[\frac{d}{2}+\frac{D}{2}-\frac{d}{2}\right]}+\frac{PL}{E\pi(\frac{D}{2}-\frac{d}{2})\frac{d}{2}}$$

$$=-\frac{PL}{E\pi(\frac{D^2}{4}-\frac{dD}{4})}+\frac{PL}{E\pi(\frac{Dd}{4}-\frac{d^2}{4})}=\frac{4PL}{E\pi}\left[-\frac{1}{D^2-dD}+\frac{1}{Dd-d^2}\right]$$

$$\delta=\frac{4PL}{\pi dDE}$$

***Example 10:*** Determine the smallest dimensions of the circular shaft and circular end cop if the load it is required to support is ***150 KN***. The allowable tensile stress, bearing stress, and shear stress is $\boldsymbol{(\sigma_t)_{allow}=175\ MPa}$, is $\boldsymbol{(\sigma_b)_{allow}=275\ MPa}$, and $\boldsymbol{\tau_{allow}=115\ MPa}$.




$(\sigma_b)_{allow}=\frac{F_b}{A_b}$

$275\times10^6=\frac{150\times10^3}{A_b}$

$A_b=0.0005454\ m^2$

$A_b=\frac{\pi}{4}d_2^2$

$d_2=\sqrt{\frac{4A_b}{\pi}}=\sqrt{\frac{4\times0.0005454}{\pi}}$

$d_2=0.026353\ m=26.353\ mm$ ⇦

$(\sigma_t)_{allow}=\frac{P}{A}$

$175\times10^6=\frac{150\times10^3}{A}$

$A=0.0008571\ m^2$

$A=\frac{\pi}{4}[d_1^2-(30\times10^{-3})^2]=0.0008571$

$d_1=0.04462\ m=44.62\ mm$ ⇦

$\tau_{allow}=\frac{F}{A}$

$115\times10^6=\frac{150\times10^3}{A}$

$A=0.0013043\ m^2$

1. $A=t\pi d$

   $0.0013043=t\times\pi\times30\times10^{-3}$

   $t=0.013839\ m=13.839\ mm$

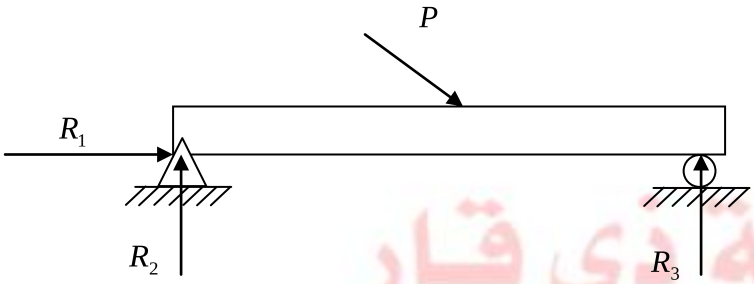
2. $A=t\pi d_2$

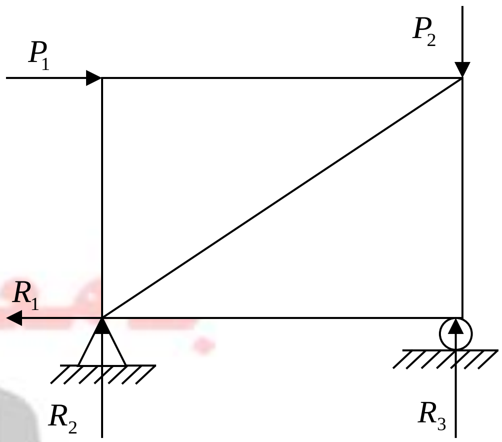
   $0.0013043=t\times\pi\times26.353\times10^{-3}$

   $t=0.01575\ m=15.75\ mm$ ⇦

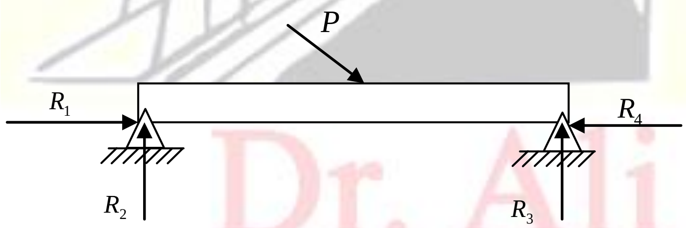
## Statically Indeterminate Members:

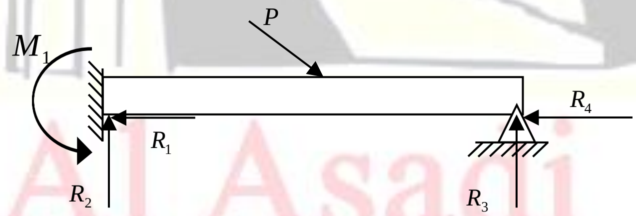
If the values of all the external forces which act on a body can be determined by the equations of static equilibrium alone, then the force system is statically determinate.



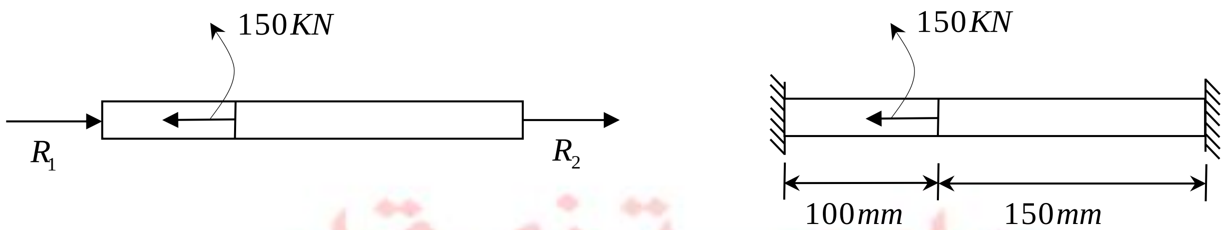




In many cases the forces acting on a body cannot be determined by the equations of static alone because there are more unknown forces than the equations of equilibrium. In such case the force system is said to be statically indeterminate.



***Example 11:*** A square bar ***50 mm*** on a side is held rigidly between the walls and loaded by an axial force of ***150 KN*** as shown. Determine the reactions at the end of the bar and the extension of the right portion. Take ***E=200 GPa***.




$R_1+R_2=150\times10^3$ ………………(1)

$\delta_1=\delta_2$

$$\frac{R_1\times100\times10^{-3}}{50\times10^{-3}\times50\times10^{-3}\times200\times10^{9}}=\frac{R_2\times150\times10^{-3}}{50\times10^{-3}\times50\times10^{-3}\times200\times10^{9}}$$

$0.1R_1=0.15R_2$

$R_1=1.5R_2$ …………………..(2)

From equations (1) and (2)

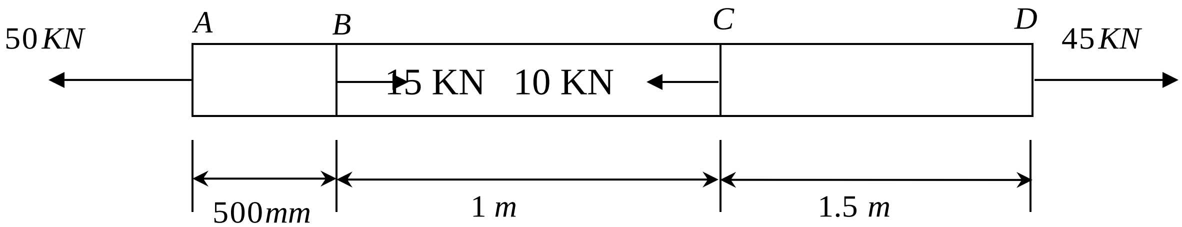
$1.5R_2+R_2=150\times10^3$

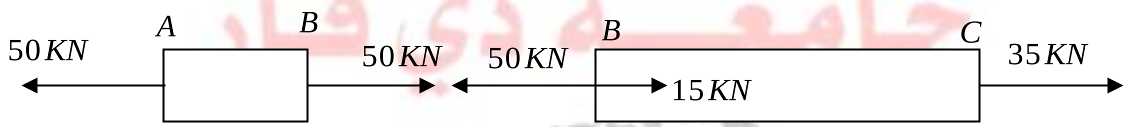
$R_2=60000$ N
$R_1=90000$ N

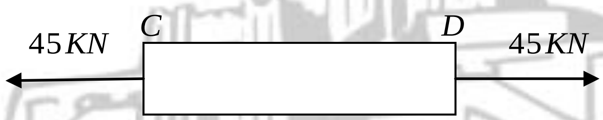
$$\delta_2=\frac{R_2\times150\times10^{-3}}{50\times10^{-3}\times50\times10^{-3}\times200\times10^{9}}=\frac{60000\times150\times10^{-3}}{50\times10^{-3}\times50\times10^{-3}\times200\times10^{9}}$$

$\delta_2=0.000018$ m
$\delta_2=0.018$ mm

***Example 12:*** A steel bar of cross section ***500 mm²*** is acted upon by the forces shown. Determine the total elongation of the bar. For steel, ***E=200 GPa***.




❖ ***For portion AB***

$$\delta_1 = \frac{PL}{AE} = \frac{50\times10^{3}\times500\times10^{-3}}{500\times10^{-6}\times200\times10^{9}} = 0.00025\ \text{m} = 0.25\ \text{mm}$$

❖ ***For portion BC***

$$\delta_2 = \frac{PL}{AE} = \frac{35\times10^{3}\times1}{500\times10^{-6}\times200\times10^{9}} = 0.00035\ \text{m} = 0.35\ \text{mm}$$

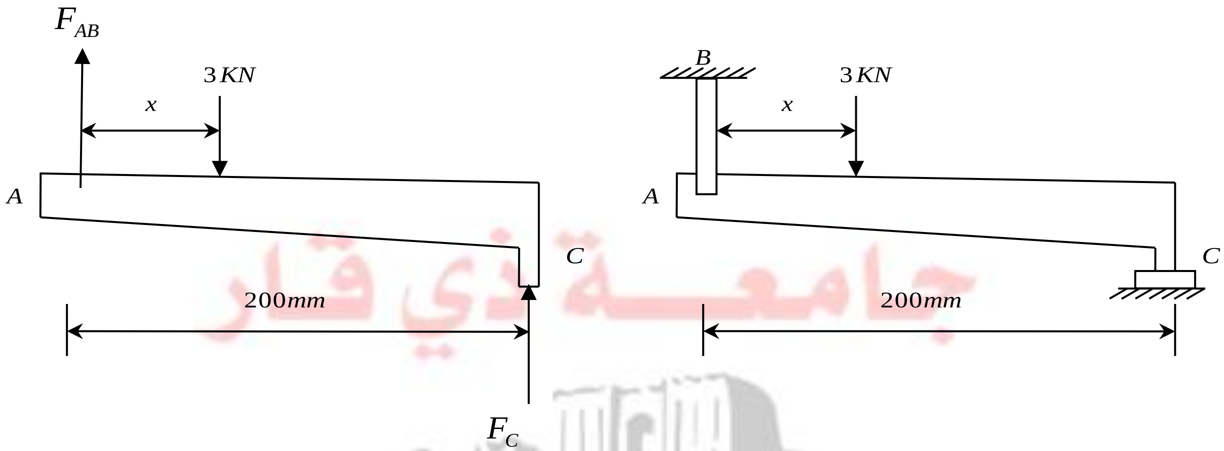
❖ ***For portion CD***

$$\delta_3 = \frac{PL}{AE} = \frac{45\times10^{3}\times1.5}{500\times10^{-6}\times200\times10^{9}} = 0.000675\ \text{m} = 0.675\ \text{mm}$$

$\delta_T = \delta_1 + \delta_2 + \delta_3$

$\delta_T = 0.25 + 0.35 + 0.675 = 1.275$ mm

***Example 13:*** Member ***AC*** shown is subjected to a vertical force of ***3 KN***. Determine the position ***x*** of this force so that the average compressive stress at ***C*** is equal to the average tensile stress in the tie rod ***AB***. The rod has a cross-sectional area of ***400 mm²*** and the contact area at ***C*** is ***650 mm²***.




$\sum F_y = 0$

$F_{AB}+F_C-3000=0$

$F_{AB}+F_C=3000$ ……………………….(1)

$\sigma_{AB}=\sigma_C$

$$\frac{F_{AB}}{A_{AB}}=\frac{F_C}{A_C}$$

$$\frac{F_{AB}}{400\times10^{-6}}=\frac{F_C}{650\times10^{-6}}$$

$F_{AB}=0.6153\ F_C$ …………………….(2)

From equations (1) and (2)

$F_C=1857.24$ N

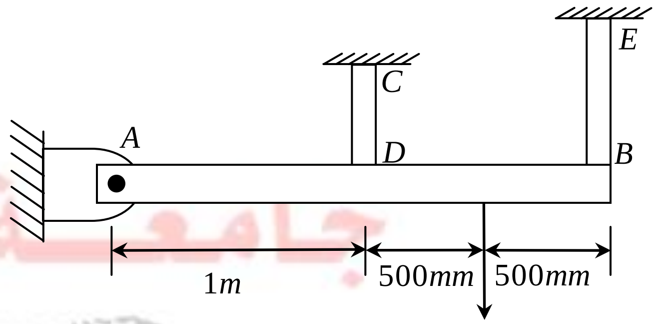
$F_{AB}=1142.759$ N

$\sum M_A = 0$

$F_C\times200\times10^{-3}-3000\times x=0$

$x=\dfrac{1857.24\times0.2}{3000}$ =0.123816 m=123.816 mm

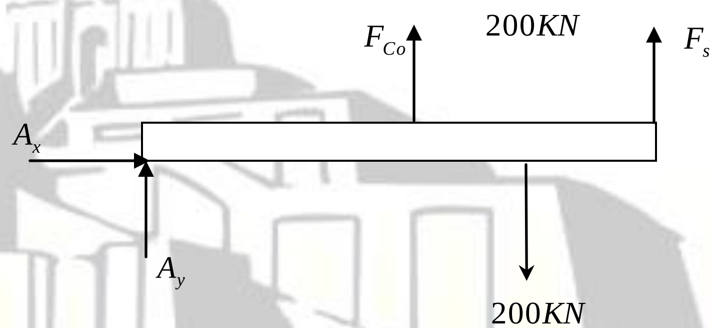
***Example 14:*** The bar ***AB*** is considered to be absolutely rigid and is horizontal before the load of ***200 KN*** is applied. The connection at ***A*** is a pin, and ***AB*** is supported by the steel rod ***EB*** and the copper rod ***CD***. The length of ***CD*** is ***1m***, of ***EB*** is ***2 m***. The cross sectional area of ***CD*** is ***500 mm²***, the area of ***EB*** is ***250 mm²***. Determine the stress in each of the vertical rods and the elongation of the steel rod. Neglect the weight of ***AB***. For copper ***E=120 GPa***, for steel ***E=200 GPa***.




$$\sum M_A = 0$$

$F_{Co}\times1+F_s\times2-200\times10^3\times1.5=0$

$F_{Co}=300\times10^3-2\,F_s$ …………..(1)



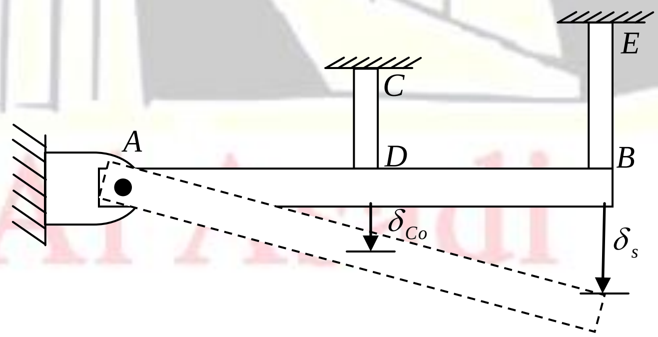

$$\frac{\delta_s}{2}=\frac{\delta_{Co}}{1}$$

$\delta_s=2\delta_{Co}$

$$\left(\frac{F_s\times L}{A_s E}\right)_s = 2\left(\frac{F_{Co}\times L}{A_{Co} E}\right)_{Co}$$

$$\frac{F_s\times2}{250\times10^{-6}\times200\times10^{9}} = 2\times\frac{F_{Co}\times1}{500\times10^{-6}\times120\times10^{9}}$$

$F_{Co}=1.2\,F_s$ …………………….(2)




From equations (1) and (2)

$F_s=93750$ N
$F_{Co}=112500$ N

$$\sigma_s=\frac{F_s}{A_s}=\frac{93750}{250\times10^{-6}}=375000000 \text{ Pa}$$

$\sigma_s=375$ MPa

$$\sigma_{Co}=\frac{F_{Co}}{A_{Co}}=\frac{112500}{500\times10^{-6}}=225000000 \text{ Pa}$$

$\sigma_{Co}=225$ MPa

$$\delta_s=\frac{\sigma_s L}{E}=\frac{375\times10^{6}\times2}{200\times10^{9}}=0.00375 \text{ m}=3.75 \text{ mm}$$

## Thermal Stresses:

A change in temperature can cause a material to change its dimensions. If the temperature increases, generally a material expands, whereas if the temperature decreases the material will contract.

The deformation of a member having a length L can be calculated using the formula:

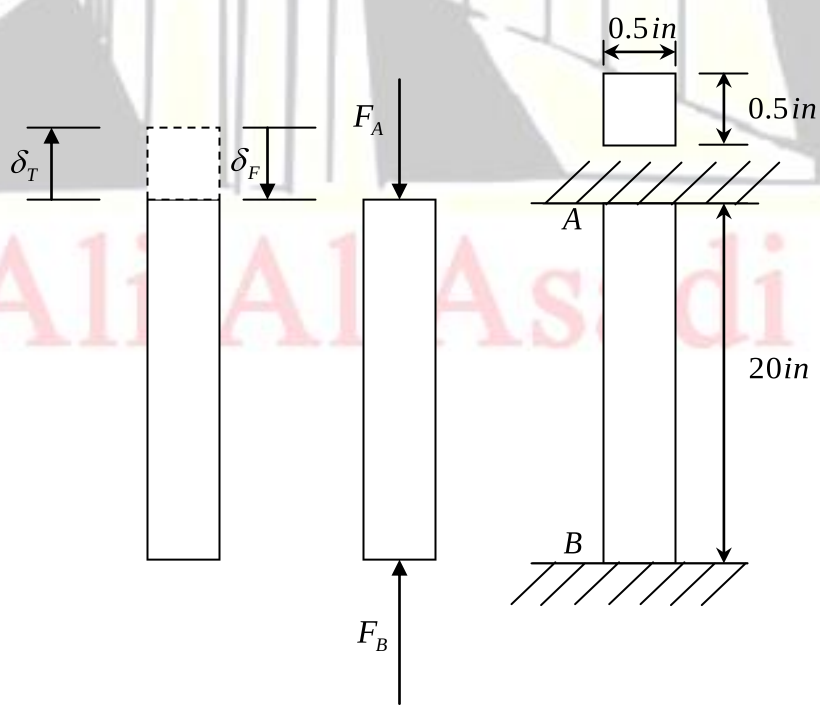
$$\delta_T = \alpha \times \Delta T \times L$$

$$\delta_T = \frac{FL}{AE} = \alpha \times \Delta T \times L$$

$$\sigma_T = E \times \alpha \times \Delta T$$

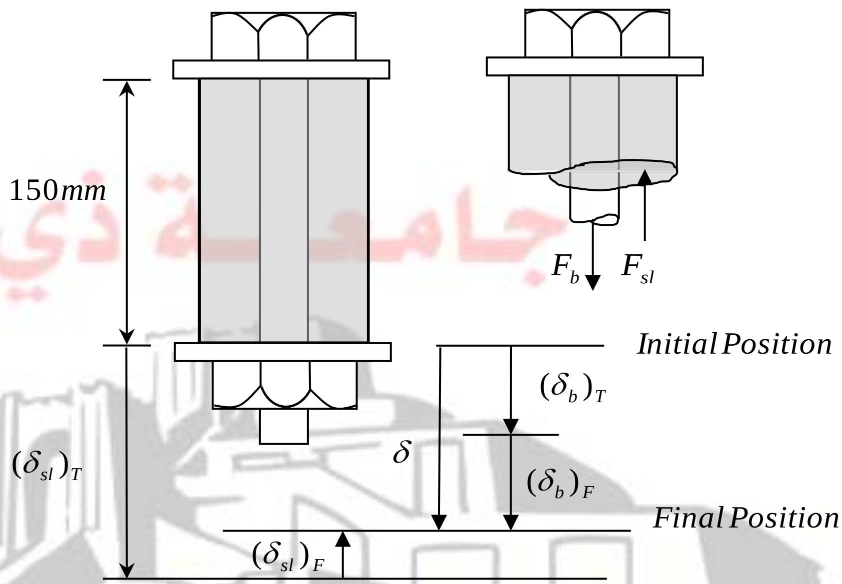
$\alpha$: Linear coefficient of thermal expansion. The units measure strain per degree of temperature. They are (1/ºF) in the foot-pound-second system and (1/ºC) or (1/ºK) in SI system.
$\Delta T$: Change in temperature of the member.
L: The original length of the member.
$\delta_T$: The change in length of the member.

***Example 15:*** The A-36 steel bar shown is constrained to just fit between two fixed supports when $T_1=60^o\ F$. If the temperature is raised to $T_2=120^o\ F$ determine the average normal thermal stress developed in the bar. For steel $\alpha=6.6\times10^{-6}\ 1/^oF$, $E=29\times10^3\ Ksi$.




$\sum F_y = 0$
$F_A - F_B = F$
$\delta_T - \delta_F = 0$
$\delta_T = \alpha \times \Delta T \times L$
$\sigma_T = E \times \alpha \times \Delta T$
$= 29\times10^3 \times 6.6\times10^{-6} \times (120-60)$
$= 11.484$ Ksi

***Example 16:*** A 2014-T6 aluminum tube having a cross sectional area of ***600 mm²*** is used as a sleeve for an A-36 steel bolt having a cross sectional area of ***400 mm²***. When the temperature is $T_1=15^o\,C$, the nut hold the assembly in a snug position such that the axial force in the bolt is negligible. If the temperature increases $T_2=80^o\,C$, determine the average normal stress in the bolt and sleeve. For aluminum $\alpha=23\times10^{-6}\ 1/^oC$, $E=73.1\ GPa$, for steel $\alpha=12\times10^{-6}\ 1/^oC$, $E=200\ GPa$.




$\sum F_y = 0$

$F_{sl}-F_b=0$

$F_{sl}=F_b=F$

$$\delta = (\delta_b)_T + (\delta_b)_F = (\delta_{sl})_T - (\delta_{sl})_F$$

$$\left[\alpha\times\Delta T\times L+\frac{FL}{AE}\right]_b=\left[\alpha\times\Delta T\times L-\frac{FL}{AE}\right]_{sl}$$

$$23\times10^{-6}\times0.15\times(80-15)-\frac{F\times0.15}{600\times10^{-6}\times73.1\times10^{9}}=12\times10^{-6}\times0.15\times(80-15)+\frac{F\times0.15}{400\times10^{-6}\times200\times10^{9}}$$

$0.0052949\times10^{-6}F=0.00010725$

$F=20255\ N$

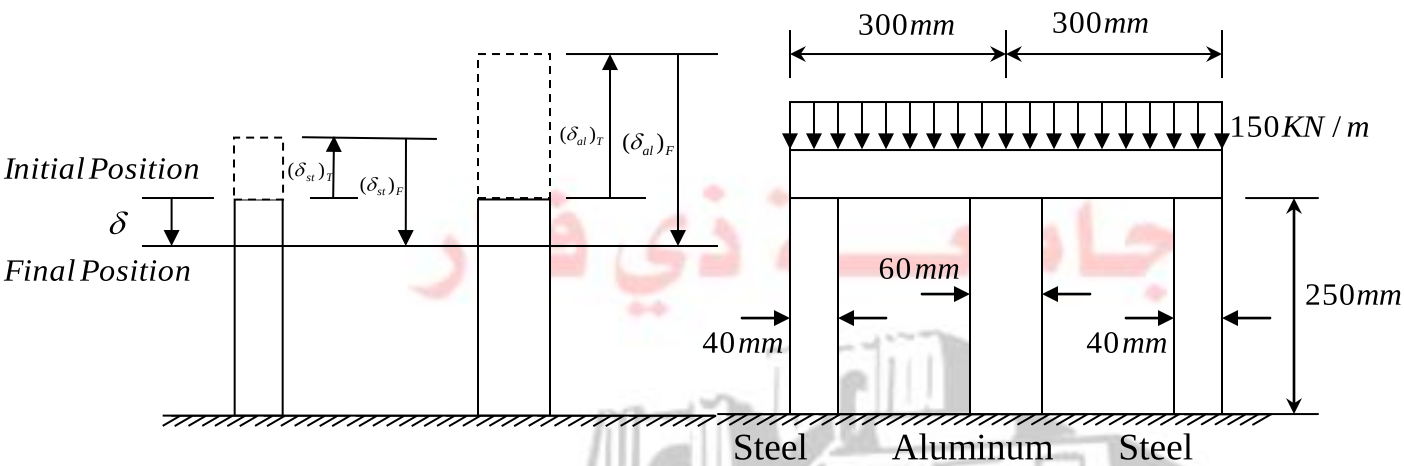
$$\sigma_b=\frac{F}{A_b}=\frac{20255}{400\times10^{-6}}$$

$\sigma_b=50.637655\ MPa$

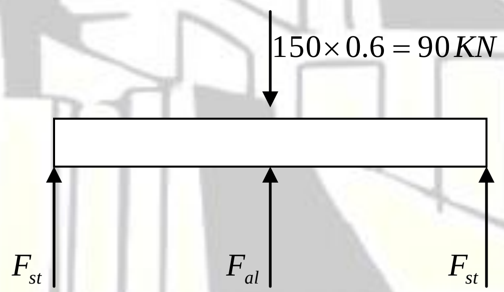
$$\sigma_{sl}=\frac{F}{A_{sl}}=\frac{20255}{600\times10^{-6}}$$

$\sigma_{sl}=33.758436\ MPa$

***Example 17:*** The rigid bar shown is fixed to the top of the three posts made of A-36 steel and 2014-T6 aluminum. The posts each have a length of ***250 mm*** when no load is applied to the bar, and the temperature is $T_1=20^oC$. Determine the force supported by each posts if the bar is subjected to a uniform distributed load of ***150 KN/m*** and the temperature is raised to $T_2=80^oC$. For steel $\alpha=12\times10^{-6}\ 1/^oC$, $E=200\ GPa$ , for aluminum $\alpha=23\times10^{-6}\ 1/^oC$, $E=73.1\ GPa$.




$\sum F_y = 0$




$2F_{st}+F_{al}=90000$ …………….(1)

$\delta=(\delta_{st})_T-(\delta_{st})_F=(\delta_{al})_T-(\delta_{al})_F$

$$[\alpha\times\Delta T\times L-\frac{F_{st}L}{AE}]_{st}=[\alpha\times\Delta T\times L-\frac{F_{al}L}{AE}]_{al}$$

$$12\times10^{-6}\times0.25\times(80-20)-\frac{F_{st}\times0.25}{\frac{\pi}{4}(40\times10^{-3})^2\times200\times10^9}=23\times10^{-6}\times0.25\times(80-20)-\frac{F_{al}\times0.25}{\frac{\pi}{4}(60\times10^{-3})^2\times73.1\times10^9}$$

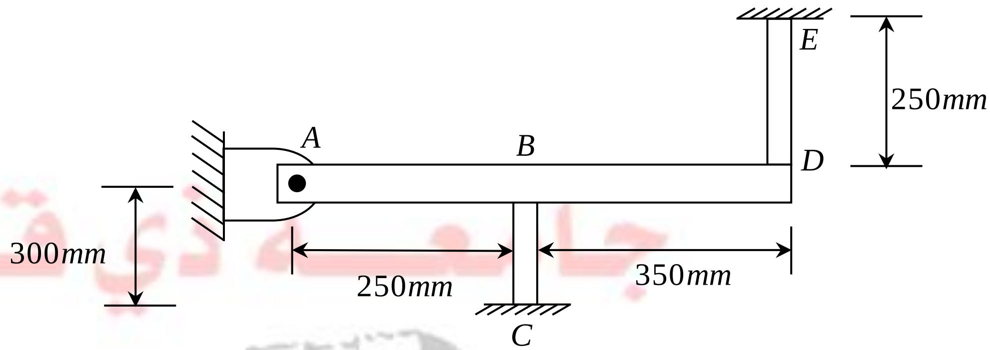
$1.20956\times10^{-9}F_{al}-0.994718\times10^{-9}F_{st}=0.000165$ ………………..(2)

From equations (1) and (2)

$F_{st}=-16444.7\ N$

$F_{al}=122888.8\ N$

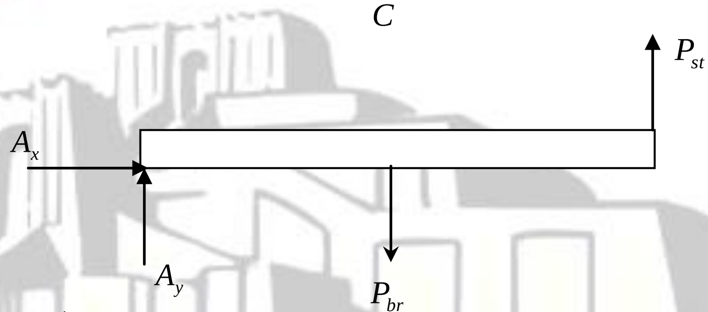
***Example 18:*** The rigid bar ***AD*** is pinned at ***A*** and attached to the bars ***BC*** and ***ED*** as shown. The entire system is initially stress-free and the weights of all bars are negligible. The temperature of bar ***BC*** is lowered ***25ºK*** and that of the bar ***ED*** is raised ***25ºK***. Neglecting any possibility of lateral buckling, find the normal stresses in bars ***BC*** and ***ED***. For ***BC***, which is brass, assume $\boldsymbol{E=90\ GPa}$, $\boldsymbol{\alpha=20\times10^{-6}\ 1/^{o}K}$ and for ***ED***, which is steel, take $\boldsymbol{\alpha=12\times10^{-6}\ 1/^{o}K}$, $\boldsymbol{E=200\ GPa}$. The cross-sectional area of ***BC*** is $\boldsymbol{500\ mm^2}$, of ***ED*** is $\boldsymbol{250\ mm^2}$.




$$\sum M_A = 0$$

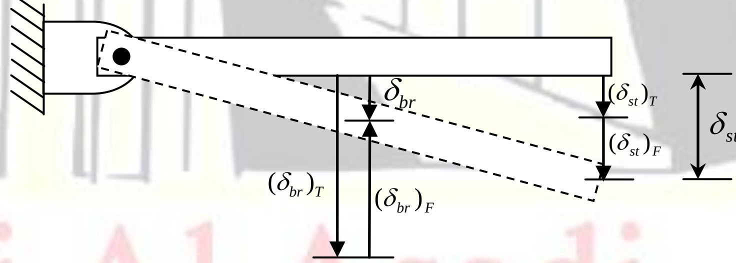
$P_{st}\times600\times10^{-3}-P_{br}\times250\times10^{-3}=0$

$P_{st}=0.41666\ P_{br}$ ………..(1)




$$\frac{\delta_{br}}{250}=\frac{\delta_{st}}{600}$$

$$\frac{\alpha\times L\times\Delta T-\dfrac{P_{br}\times L}{A_{br}E_{br}}}{250}=\frac{\alpha\times L\times\Delta T+\dfrac{P_{st}\times L}{A_{st}E_{st}}}{600}$$




$$\frac{20\times10^{-6}\times300\times10^{-3}\times25-\dfrac{P_{br}\times300\times10^{-3}}{500\times10^{-6}\times90\times10^{9}}}{250}=\frac{12\times10^{-6}\times250\times10^{-3}\times25+\dfrac{P_{st}\times250\times10^{-3}}{250\times10^{-6}\times200\times10^{9}}}{600}$$

$8.333\times10^{-12}\ P_{st}+26.666\times10^{-12}\ P_{br}=475\times10^{-9}$ …………..(2)

From equations (1) and (2)
$P_{br}=15760.5$ N , $P_{st}=6566.77$ N

$$\sigma_{br}=\frac{15760.5}{500\times10^{-6}}=31.521MPa$$

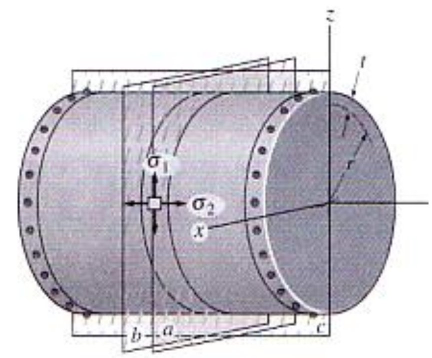
$$\sigma_{st}=\frac{6566.77}{250\times10^{-6}}=26.267MPa$$

## Thin Walled Cylinder, Thin Walled Pressure Vessels:

Cylindrical or spherical vessels are commonly used in industry to serve as boilers or tanks. When under pressure, the material of which they are made is subjected to a loading from all directions. In general "thin wall" refers to a vessel having an inner radius to wall thickness ratio of 10 or more ($r/t \geq 10$)

### 1. Cylindrical Vessels:

Consider the cylindrical vessel having a wall thickness t and inner radius r as shown below. A pressure p is developed within the vessel by a containing gas or fluid, which is assumed to have negligible weight.

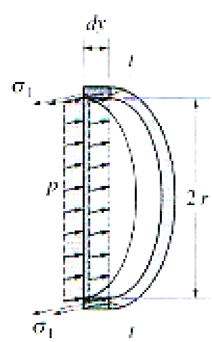



The stresses set up in the walls are:

**a. Circumferential or hoop stress**

$$2[\sigma_1(tdy)]-p(2rdy)=0$$

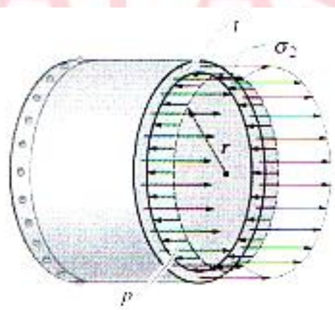
$$\sigma_1 = \frac{pr}{t}$$




**b. Longitudinal or axial stress**

$$\sigma_2(2\pi rt)-p(\pi r^2)=0$$

$$\sigma_2 = \frac{pr}{2t}$$



**c. Circumferential or hoop strain**

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2)$$

**d. Longitudinal strain**

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1)$$

**e. Change in length**

The change in length of the cylinder may be determined from the longitudinal strain.
Change in length=longitudinal strain×original length

$$\delta L = \varepsilon_2 L = \frac{1}{E}(\sigma_2 - \nu\sigma_1)\,L$$

$$\delta L = \frac{pr}{2tE}(1-2\nu)\,L$$

**f. Change in diameter**

The change in diameter may be found from the circumferential change.
Change in diameter=diametral strain×original diameter
Diametral strain=circumferential strain

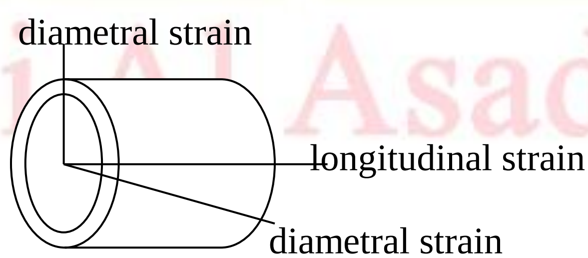
$$\delta d = \varepsilon_1 d = \frac{1}{E}(\sigma_1 - \nu\sigma_2)\,d$$

$$\delta d = \frac{pr}{2tE}(2-\nu)\,d$$

**g. Change in internal volume**

Volumetric strain=longitudinal strain+2diametral strain

$$\varepsilon_v = \varepsilon_2 + 2\,\varepsilon_1 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) + 2\frac{1}{E}(\sigma_1 - \nu\sigma_2)$$




$$\varepsilon_v = \frac{1}{E}(\sigma_2 - \nu\sigma_1 + 2\sigma_1 - 2\nu\sigma_2)$$

$$= \frac{1}{E}\left(\frac{pr}{2t} - \nu\frac{pr}{t} + 2\frac{pr}{t} - \nu\frac{pr}{t}\right)$$

$$\varepsilon_v = \frac{pr}{2tE}(5-4\nu)$$

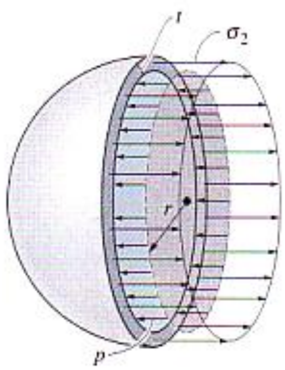
change in internal volume=volumetric strain×original volume

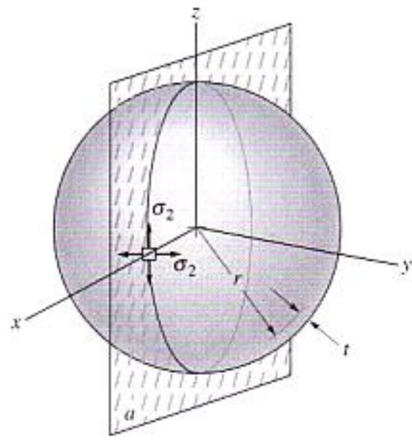
$$\delta v = \varepsilon_v v$$

$$\delta v = \frac{pr}{2tE}(5-4\nu)\,v$$

## 2. Spherical Vessels:

Because of the symmetry of the sphere the stresses set up owing to internal pressure will be two mutually perpendicular hoop or circumferential stress of equal value and a radial stress.




$\sigma_1(2\pi rt)-p(\pi r^2)=0$

$$\sigma_1 = \frac{pr}{2t}$$

$$\sigma_2 = \sigma_1 = \frac{pr}{2t}$$

**Change in internal volume**

change in internal volume=volumetric strain×original volume
volumetric strain=3hoop strain

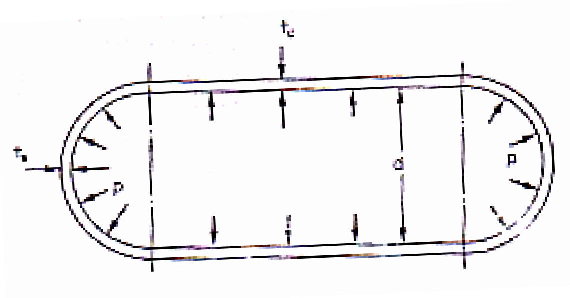
$$\varepsilon_v = \varepsilon_1 = 3\frac{1}{E}(\sigma_1 - \nu\sigma_2) = \frac{3\sigma_1}{E}(1-\nu) = \frac{3pr}{2tE}(1-\nu)$$

$$\delta v = \varepsilon_v v$$

$$\delta v = \frac{3pr}{2tE}(1-\nu)\, v$$

## Cylindrical Vessels with Hemispherical Ends:

r=d/2




a) For the cylindrical portion

$$\sigma_1 = \frac{\Pr}{t_c} \quad \text{hoop stress}$$

$$\sigma_2 = \frac{\Pr}{2t_c} \quad \text{longitudinal stress}$$

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) = \frac{1}{E}\left(\frac{\Pr}{t_c} - \nu\frac{\Pr}{2t_c}\right)$$

$$\varepsilon_1 = \frac{pr}{2t_c E}(2-\nu) \qquad \text{hoop strain}$$

b) For the spherical ends

$$\sigma_1 = \frac{\Pr}{2t_s} \qquad \text{hoop stress}$$

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) = \frac{\sigma_1}{E}(1-\nu)$$

$$\varepsilon_1 = \frac{pr}{2t_s E}(1-\nu) \qquad \text{hoop strain}$$

Thus equating the two strains in order that there shall be no distortion of the junction.

$$\frac{pr}{2t_s E}(1-\nu) = \frac{pr}{2t_c E}(2-\nu)$$

$$\frac{t_s}{t_c} = \frac{1-\upsilon}{2-\upsilon}$$

***Example 30:*** A thin cylinder ***75 mm*** internal diameter, ***250 mm*** long with walls ***2.5 mm*** thick is subjected to an internal pressure of ***7 MN/m²***. Determine the change in internal diameter and the change in length. If in addition to the internal pressure, the cylinder is subjected to a torque of ***200 N.m*** find the magnitude and nature of the stresses set up in the cylinder. ***E=200 GN/m², υ=0.3***.

$$\delta d=\frac{pr}{2tE}(2-\nu)\,d$$

$$\delta d=\frac{7\times10^{6}\times\frac{75}{2}\times10^{-3}}{2\times2.5\times10^{-3}\times200\times10^{9}}[2-0.3]\times75\times10^{-3}$$

$$\delta d=33.468\times10^{-6}\ m=33.468\ \mu m$$

$$\delta L=\frac{pr}{2tE}(1-2\nu)\,L$$

$$\delta L=\frac{7\times10^{6}\times\frac{75}{2}\times10^{-3}}{2\times2.5\times10^{-3}\times200\times10^{9}}[1-2\times0.3]\times250\times10^{-3}$$

$$\delta L=26.25\times10^{-6}\ m=26.25\ \mu m$$

$$\sigma_1=\frac{pr}{t}=\frac{7\times10^{6}\times\frac{75}{2}\times10^{-3}}{2.5\times10^{-3}}$$

$$\sigma_1=105\times10^{6}\ N/m^2=105\ MN/m^2$$

$$\sigma_2=\frac{pr}{2t}=\frac{7\times10^{6}\times\frac{75}{2}\times10^{-3}}{2\times2.5\times10^{-3}}$$

$$\sigma_2=52.5\times10^{6}\ N/m^2=52.5\ MN/m^2$$

$$\tau=\frac{Tr}{J}=\frac{Tr}{\frac{\pi}{2}[r_o^4-r_i^4]}=\frac{200\times40\times10^{-3}}{\frac{\pi}{2}[(40\times10^{-3})^4-(37.5\times10^{-3})^4]}$$

$$\tau=8.743862\ MN/m^2$$

***Example 31:*** A cylinder has an internal diameter of ***230 mm***, has walls ***5 mm*** thick and is ***1 m*** long. It is found to change in internal volume by ***$12\times10^{-6}$ $m^3$*** when filled with a liquid at a pressure ***p***. If ***$E=200\ GN/m^2$*** and ***$\upsilon=0.25$***, and assuming rigid end plates, determine ***a)*** the values of hoop and longitudinal stresses ***b)*** the necessary change in pressure ***p*** to produce a further increase in internal volume of ***15%***.

a) $\delta v=\frac{pr}{2tE}(5-4\nu)\,v$

$$12\times10^{-6}=\frac{p\times\frac{230}{2}\times10^{-3}}{2\times5\times10^{-3}\times200\times10^{9}}[5-4\times0.25]\times\pi\times(\frac{230}{2}\times10^{-3})^2\times1$$

$p=1.255763\ MN/m^2$

$$\sigma_1=\frac{pr}{t}=\frac{1.255763\times10^{6}\times\frac{230}{2}\times10^{-3}}{5\times10^{-3}}$$

$\sigma_1=28.882549\ MN/m^2$

$$\sigma_2=\frac{pr}{2t}=\frac{1.255763\times10^{6}\times\frac{230}{2}\times10^{-3}}{2\times5\times10^{-3}}$$

$\sigma_2=14.4412745\ MN/m^2$

b) $\delta v=1.15\times12\times10^{-6}=13.8\times10^{-6}\ m^3$

$\delta v=\frac{pr}{2tE}(5-4\nu)\,v$

$$13.8\times10^{-6}=\frac{p\times\frac{230}{2}\times10^{-3}}{2\times5\times10^{-3}\times200\times10^{9}}[5-4\times0.25]\times\pi\times(\frac{230}{2}\times10^{-3})^2\times1$$

$p=1.444128\ MN/m^2$

Necessary increase=1.444128-1.255763=0.188365 $MN/m^2$

## Vessels Subjected to Fluid Pressure:

If a fluid is used as the pressurization medium the fluid itself will change in volume as pressure is increased and this must be taken into account when calculating the amount of fluid which must be pumped into the cylinder in order to raise the pressure by a specific amount.

The bulk modulus of a fluid is defined as:

bulk modulus $k=\dfrac{Volumetric stress}{Volumetric strain}$

volumetric stress=pressure p

volumetric strain=$\dfrac{change in volume}{original volume}=\dfrac{\delta v}{v}$

$k=\dfrac{p}{\frac{\delta v}{v}}=\dfrac{pv}{\delta v}$

change in volume of fluid under pressure=$\dfrac{pv}{k}$

extra fluid required to raise cylinder pressure by p

$=\dfrac{pr}{2tE}(5-4\nu)\,\text{v}+\dfrac{pv}{k}$

extra fluid required to raise sphere pressure by p

$=\dfrac{3pr}{2tE}(1-\nu)\,\text{v}+\dfrac{pv}{k}$

***Example 32: a)*** A sphere 1m internal diameter and ***6 mm*** wall thickness is to be pressure tested for safety purposes with water as the pressure medium. Assuming that the sphere is initially filled with water at atmospheric pressure, what extra volume of water is required to be pumped in to produce a pressure of ***3 MN/m²*** gauge? For water ***k=2.1 GN/m²*** ***b)*** The sphere is now placed in service and filled with gas until there is a volume change of ***72×10⁻⁶ m³***. Determine the pressure exerted by the gas on the walls of the sphere. ***c)*** To what value can the gas pressure be increased before failure occurs according to the maximum principal stress theory of elastic failure? ***E=200 GPa***, ***υ=0.3*** and the yield stress is simple tension***=280 MPa***.

a) extra volume of water$=\frac{3pr}{2tE}(1-\nu)\,\text{v}+\frac{p\text{v}}{k}$

$$=\frac{3\times 3\times 10^{6}\times 0.5}{2\times 6\times 10^{-3}\times 200\times 10^{9}}(1-0.3)\times\frac{4}{3}\pi(0.5)^{3}+\frac{3\times 10^{6}\times\frac{4}{3}\pi(0.5)^{3}}{2.1\times 10^{9}}$$

$=0.001435221\ \text{m}^3$

b) $\delta\text{v}=\frac{3pr}{2tE}(1-\nu)\,\text{v}$

$$72\times 10^{-6}=\frac{3p\times 0.5}{2\times 6\times 10^{-3}\times 200\times 10^{9}}(1-0.3)\times\frac{4}{3}\pi(0.5)^{3}$$

$p=0.31430827\ \text{MN/m}^2$

$\sigma_1=\frac{pr}{2t}$ $\sigma_1$=yield stress for maximum principal stress theory

$$280\times 10^{6}=\frac{p\times 0.5}{2\times 6\times 10^{-3}}$$

$p=6.72\ \text{MN/m}^2$