Chapter 48

Standard integration

48.1 The process of integration

The process of integration reverses the process of differentiation. In differentiation, if $f(x) = 2x^2$ then $f'(x) = 4x$. Thus the integral of $4x$ is $2x^2$, i.e. integration is the process of moving from $f'(x)$ to $f(x)$. By similar reasoning, the integral of $2t$ is t^2.

Integration is a process of summation or adding parts together and an elongated S, shown as $\int$, is used to replace the words 'the integral of'. Hence, from above, $\int 4x = 2x^2$ and $\int 2t$ is t^2.

In differentiation, the differential coefficient $\frac{dy}{dx}$ indicates that a function of x is being differentiated with respect to x, the dx indicating that it is 'with respect to x'. In integration the variable of integration is shown by adding d(the variable) after the function to be integrated.

Thus $\int 4x\,dx$ means 'the integral of $4x$ with respect to x',

and $\int 2t\,dt$ means 'the integral of $2t$ with respect to t'

As stated above, the differential coefficient of $2x^2$ is $4x$, hence $\int 4x\,dx = 2x^2$. However, the differential coefficient of $2x^2 + 7$ is also $4x$. Hence $\int 4x dx$ is also equal to $2x^2 + 7$. To allow for the possible presence of a constant, whenever the process of integration is performed, a constant 'c' is added to the result.

Thus $\int 4x\,dx = 2x^2 + c$ and $\int 2t\,dt = t^2 + c$

'c' is called the **arbitrary constant of integration.**

48.2 The general solution of integrals of the form ax^n

The general solution of integrals of the form $\int ax^n dx$, where a and n are constants is given by:

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$$

This rule is true when n is fractional, zero, or a positive or negative integer, with the exception of $n = -1$.
Using this rule gives:

(i) $\int 3x^4 dx = \frac{3x^{4+1}}{4+1} + c = \mathbf{\frac{3}{5}x^5 + c}$

(ii) $\int \frac{2}{x^2} dx = \int 2x^{-2} dx = \frac{2x^{-2+1}}{-2+1} + c$

$= \frac{2x^{-1}}{-1} + c = \mathbf{\frac{-2}{x} + c}$, and

(iii) $\int \sqrt{x}\,dx = \int x^{1/2} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \mathbf{\frac{2}{3}\sqrt{x^3} + c}$

Each of these three results may be checked by differentiation.

(a) The integral of a constant k is $kx + c$. For example,

$$\int 8\,dx = 8x + c$$

(b) When a sum of several terms is integrated the result is the sum of the integrals of the separate terms. For example,

$$\int (3x + 2x^2 - 5)dx$$
$$= \int 3x\,dx + \int 2x^2 dx - \int 5\,dx$$
$$= \mathbf{\frac{3x^2}{2} + \frac{2x^3}{3} - 5x + c}$$

48.3 Standard integrals

Since integration is the reverse process of differentiation the **standard integrals** listed in Table 48.1 may be deduced and readily checked by differentiation.

Table 48.1 Standard integrals

(i)	$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$ (except when $n = -1$)
(ii)	$\int \cos ax\,dx = \frac{1}{a}\sin ax + c$
(iii)	$\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$
(iv)	$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c$
(v)	$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + c$
(vi)	$\int \operatorname{cosec} ax \cot ax\,dx = -\frac{1}{a}\operatorname{cosec} ax + c$
(vii)	$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + c$
(viii)	$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
(ix)	$\int \frac{1}{x} dx = \ln x + c$

Problem 1. Determine:

(a) $\int 5x^2 dx$ (b) $\int 2t^3 dt$

The standard integral, $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$

(a) When $a = 5$ and $n = 2$ then

$$\int 5x^2 dx = \frac{5x^{2+1}}{2+1} + c = \mathbf{\frac{5x^3}{3} + c}$$

(b) When $a = 2$ and $n = 3$ then

$$\int 2t^3 dt = \frac{2t^{3+1}}{3+1} + c = \frac{2t^4}{4} + c = \mathbf{\frac{1}{2}t^4 + c}$$

Each of these results may be checked by differentiating them.

Problem 2. Determine $\int \left(4 + \frac{3}{7}x - 6x^2\right) dx$

$\int \left(4 + \frac{3}{7}x - 6x^2\right) dx$ may be written as

$\int 4\,dx + \int \frac{3}{7}x\,dx - \int 6x^2\,dx$

i.e. each term is integrated separately. (This splitting up of terms only applies, however, for addition and subtraction).

Hence $\int \left(4 + \frac{3}{7}x - 6x^2\right)dx$

$$= 4x + \left(\frac{3}{7}\right)\frac{x^{1+1}}{1+1} - (6)\frac{x^{2+1}}{2+1} + c$$
$$= 4x + \left(\frac{3}{7}\right)\frac{x^2}{2} - (6)\frac{x^3}{3} + c$$
$$= \mathbf{4x + \frac{3}{14}x^2 - 2x^3 + c}$$

Note that when an integral contains more than one term there is no need to have an arbitrary constant for each; just a single constant at the end is sufficient.

Problem 3. Determine

(a) $\int \frac{2x^3 - 3x}{4x} dx$ (b) $\int (1-t)^2 dt$

(a) Rearranging into standard integral form gives:

$$\int \frac{2x^3 - 3x}{4x} dx = \int \frac{2x^3}{4x} - \frac{3x}{4x} dx$$
$$= \int \frac{x^2}{2} - \frac{3}{4} dx = \left(\frac{1}{2}\right)\frac{x^{2+1}}{2+1} - \frac{3}{4}x + c$$
$$= \left(\frac{1}{2}\right)\frac{x^3}{3} - \frac{3}{4}x + c = \mathbf{\frac{1}{6}x^3 - \frac{3}{4}x + c}$$

(b) Rearranging $\int (1-t)^2 dt$ gives:

$$\int (1-2t+t^2)dt = t - \frac{2t^{1+1}}{1+1} + \frac{t^{2+1}}{2+1} + c$$

$$= t - \frac{2t^2}{2} + \frac{t^3}{3} + c$$

$$= \mathbf{t - t^2 + \frac{1}{3}t^3 + c}$$

This problem shows that functions often have to be rearranged into the standard form of $\int ax^n dx$ before it is possible to integrate them.

Problem 4. Determine $\int \frac{3}{x^2}dx$

$\int \frac{3}{x^2}dx = \int 3x^{-2}$. Using the standard integral, $\int ax^n dx$ when $a=3$ and $n=-2$ gives:

$$\int 3x^{-2}dx = \frac{3x^{-2+1}}{-2+1} + c = \frac{3x^{-1}}{-1} + c$$

$$= -3x^{-1} + c = \mathbf{\frac{-3}{x} + c}$$

Problem 5. Determine $\int 3\sqrt{x}dx$

For fractional powers it is necessary to appreciate $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

$$\int 3\sqrt{x}dx = \int 3x^{1/2}dx = \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$= \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + c = 2x^{\frac{3}{2}} + c = \mathbf{2\sqrt{x^3} + c}$$

Problem 6. Determine $\int \frac{-5}{9\sqrt[4]{t^3}}dt$

$$\int \frac{-5}{9\sqrt[4]{t^3}}dt = \int \frac{-5}{9t^{\frac{3}{4}}}dt = \int \left(-\frac{5}{9}\right)t^{-\frac{3}{4}}dt$$

$$= \left(-\frac{5}{9}\right)\frac{t^{-\frac{3}{4}+1}}{-\frac{3}{4}+1} + c$$

$$= \left(-\frac{5}{9}\right)\frac{t^{\frac{1}{4}}}{\frac{1}{4}} + c = \left(-\frac{5}{9}\right)\left(\frac{4}{1}\right)t^{1/4} + c$$

$$= \mathbf{-\frac{20}{9}\sqrt[4]{t} + c}$$

Problem 7. Determine $\int \frac{(1+\theta)^2}{\sqrt{\theta}}d\theta$

$$\int \frac{(1+\theta)^2}{\sqrt{\theta}}d\theta = \int \frac{(1+2\theta+\theta^2)}{\sqrt{\theta}}d\theta$$

$$= \int \left(\frac{1}{\theta^{\frac{1}{2}}} + \frac{2\theta}{\theta^{\frac{1}{2}}} + \frac{\theta^2}{\theta^{\frac{1}{2}}}\right)d\theta$$

$$= \int \left(\theta^{-\frac{1}{2}} + 2\theta^{1-\left(\frac{1}{2}\right)} + \theta^{2-\left(\frac{1}{2}\right)}\right)d\theta$$

$$= \int \left(\theta^{-\frac{1}{2}} + 2\theta^{\frac{1}{2}} + \theta^{\frac{3}{2}}\right)d\theta$$

$$= \frac{\theta^{\left(-\frac{1}{2}\right)+1}}{-\frac{1}{2}+1} + \frac{2\theta^{\left(\frac{1}{2}\right)+1}}{\frac{1}{2}+1} + \frac{\theta^{\left(\frac{3}{2}\right)+1}}{\frac{3}{2}+1} + c$$

$$= \frac{\theta^{\frac{1}{2}}}{\frac{1}{2}} + \frac{2\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{\theta^{\frac{5}{2}}}{\frac{5}{2}} + c$$

$$= 2\theta^{\frac{1}{2}} + \frac{4}{3}\theta^{\frac{3}{2}} + \frac{2}{5}\theta^{\frac{5}{2}} + c$$

$$= \mathbf{2\sqrt{\theta} + \frac{4}{3}\sqrt{\theta^3} + \frac{2}{5}\sqrt{\theta^5} + c}$$

Problem 8. Determine

(a) $\int 4\cos 3x\, dx$ (b) $\int 5\sin 2\theta\, d\theta$

(a) From Table 48.1 (ii),

$$\int 4\cos 3x\, dx = (4)\left(\frac{1}{3}\right)\sin 3x + c$$

$$= \mathbf{\frac{4}{3}\sin 3x + c}$$

(b) From Table 48.1(iii),

$$\int 5\sin 2\theta\, d\theta = (5)\left(-\frac{1}{2}\right)\cos 2\theta + c$$

$$= \mathbf{-\frac{5}{2}\cos 2\theta + c}$$

Problem 9. Determine (a) $\int 7\sec^2 4t\,dt$ (b) $3\int \operatorname{cosec}^2 2\theta\,d\theta$

(a) From Table 48.1(iv),

$$\int 7\sec^2 4t\,dt = (7)\left(\frac{1}{4}\right)\tan 4t + c = \mathbf{\frac{7}{4}\tan 4t + c}$$

(b) From Table 48.1(v),

$$3\int \operatorname{cosec}^2 2\theta\,d\theta = (3)\left(-\frac{1}{2}\right)\cot 2\theta + c = \mathbf{-\frac{3}{2}\cot 2\theta + c}$$

Problem 10. Determine (a) $\int 5e^{3x}dx$ (b) $\int \frac{2}{3e^{4t}}dt$

(a) From Table 48.1(viii),

$$\int 5e^{3x}dx = (5)\left(\frac{1}{3}\right)e^{3x} + c = \mathbf{\frac{5}{3}e^{3x} + c}$$

(b)
$$\int \frac{2}{3e^{4t}}dt = \int \frac{2}{3}e^{-4t}dt = \left(\frac{2}{3}\right)\left(-\frac{1}{4}\right)e^{-4t} + c = -\frac{1}{6}e^{-4t} + c = \mathbf{-\frac{1}{6e^{4t}} + c}$$

Problem 11. Determine (a) $\int \frac{3}{5x}dx$ (b) $\int \left(\frac{2m^2+1}{m}\right)dm$

(a) $$\int \frac{3}{5x}dx = \int \left(\frac{3}{5}\right)\left(\frac{1}{x}\right)dx = \mathbf{\frac{3}{5}\ln x + c}$$

(from Table 48.1(ix))

(b)
$$\int \left(\frac{2m^2+1}{m}\right)dm = \int \left(\frac{2m^2}{m} + \frac{1}{m}\right)dm = \int \left(2m + \frac{1}{m}\right)dm = \frac{2m^2}{2} + \ln m + c = \mathbf{m^2 + \ln m + c}$$

Now try the following exercise

Exercise 172 Further problems on standard integrals

Determine the following integrals:

1. (a) $\int 4\,dx$ (b) $\int 7x\,dx$
   $\left[\text{(a) } 4x + c \quad \text{(b) } \frac{7x^2}{2} + c\right]$

2. (a) $\int \frac{2}{5}x^2dx$ (b) $\int \frac{5}{6}x^3dx$
   $\left[\text{(a) } \frac{2}{15}x^3 + c \quad \text{(b) } \frac{5}{24}x^4 + c\right]$

3. (a) $\int \left(\frac{3x^2 - 5x}{x}\right)dx$ (b) $\int (2+\theta)^2 d\theta$
   $\left[\text{(a) } \frac{3x^2}{2} - 5x + c \quad \text{(b) } 4\theta + 2\theta^2 + \frac{\theta^3}{3} + c\right]$

4. (a) $\int \frac{4}{3x^2}dx$ (b) $\int \frac{3}{4x^4}dx$
   $\left[\text{(a) } \frac{-4}{3x} + c \quad \text{(b) } \frac{-1}{4x^3} + c\right]$

5. (a) $2\int \sqrt{x^3}dx$ (b) $\int \frac{1}{4}\sqrt[4]{x^5}dx$
   $\left[\text{(a) } \frac{4}{5}\sqrt{x^5} + c \quad \text{(b) } \frac{1}{9}\sqrt[4]{x^9} + c\right]$

6. (a) $\int \frac{-5}{\sqrt{t^3}}dt$ (b) $\int \frac{3}{7\sqrt[5]{x^4}}dx$
   $\left[\text{(a) } \frac{10}{\sqrt{t}} + c \quad \text{(b) } \frac{15}{7}\sqrt[5]{x} + c\right]$

7. (a) $\int 3\cos 2x\,dx$ (b) $\int 7\sin 3\theta d\theta$
   $\left[\text{(a) } \frac{3}{2}\sin 2x + c \quad \text{(b) } -\frac{7}{3}\cos 3\theta + c\right]$

8. (a) $\int \frac{3}{4}\sec^2 3x\,dx$ (b) $\int 2\operatorname{cosec}^2 4\theta\,d\theta$
   $\left[\text{(a) } \frac{1}{4}\tan 3x + c \quad \text{(b) } -\frac{1}{2}\cot 4\theta + c\right]$

Section 9

9. (a) $5\int \cot 2t \operatorname{cosec} 2t\, dt$

(b) $\int \frac{4}{3} \sec 4t \tan 4t\, dt$

$\left[(a)\ -\frac{5}{2}\operatorname{cosec} 2t + c \quad (b)\ \frac{1}{3}\sec 4t + c\right]$

10. (a) $\int \frac{3}{4} e^{2x} dx$ (b) $\frac{2}{3}\int \frac{dx}{e^{5x}}$

$\left[(a)\ \frac{3}{8}e^{2x} + c \quad (b)\ \frac{-2}{15e^{5x}} + c\right]$

11. (a) $\int \frac{2}{3x} dx$ (b) $\int \left(\frac{u^2-1}{u}\right) du$

$\left[(a)\ \frac{2}{3}\ln x + c \quad (b)\ \frac{u^2}{2} - \ln u + c\right]$

12. (a) $\int \frac{(2+3x)^2}{\sqrt{x}} dx$ (b) $\int \left(\frac{1}{t} + 2t\right)^2 dt$

$\left[\begin{array}{l}(a)\ 8\sqrt{x} + 8\sqrt{x^3} + \frac{18}{5}\sqrt{x^5} + c \\ (b)\ -\frac{1}{t} + 4t + \frac{4t^3}{3} + c\end{array}\right]$

48.4 Definite integrals

Integrals containing an arbitrary constant c in their results are called **indefinite integrals** since their precise value cannot be determined without further information. **Definite integrals** are those in which limits are applied. If an expression is written as $[x]_a^b$, 'b' is called the upper limit and 'a' the lower limit.

The operation of applying the limits is defined as: $[x]_a^b = (b) - (a)$

The increase in the value of the integral x^2 as x increases from 1 to 3 is written as $\int_1^3 x^2\, dx$

Applying the limits gives:

$$\int_1^3 x^2 dx = \left[\frac{x^3}{3} + c\right]_1^3 = \left(\frac{3^3}{3} + c\right) - \left(\frac{1^3}{3} + c\right)$$

$$= (9 + c) - \left(\frac{1}{3} + c\right) = \mathbf{8\frac{2}{3}}$$

Note that the 'c' term always cancels out when limits are applied and it need not be shown with definite integrals.

Problem 12. Evaluate (a) $\int_1^2 3x\, dx$ (b) $\int_{-2}^3 (4 - x^2) dx$

(a) $\int_1^2 3x\, dx = \left[\frac{3x^2}{2}\right]_1^2 = \left\{\frac{3}{2}(2)^2\right\} - \left\{\frac{3}{2}(1)^2\right\}$

$$= 6 - 1\frac{1}{2} = \mathbf{4\frac{1}{2}}$$

(b) $\int_{-2}^3 (4 - x^2) dx = \left[4x - \frac{x^3}{3}\right]_{-2}^3$

$$= \left\{4(3) - \frac{(3)^3}{3}\right\} - \left\{4(-2) - \frac{(-2)^3}{3}\right\}$$

$$= \{12 - 9\} - \left\{-8 - \frac{-8}{3}\right\}$$

$$= \{3\} - \left\{-5\frac{1}{3}\right\} = \mathbf{8\frac{1}{3}}$$

Problem 13. Evaluate $\int_1^4 \left(\frac{\theta + 2}{\sqrt{\theta}}\right) d\theta$, taking positive square roots only

$$\int_1^4 \left(\frac{\theta + 2}{\sqrt{\theta}}\right) d\theta = \int_1^4 \left(\frac{\theta}{\theta^{\frac{1}{2}}} + \frac{2}{\theta^{\frac{1}{2}}}\right) d\theta$$

$$= \int_1^4 \left(\theta^{\frac{1}{2}} + 2\theta^{-\frac{1}{2}}\right) d\theta$$

$$= \left[\frac{\theta^{\left(\frac{1}{2}\right)+1}}{\frac{1}{2}+1} + \frac{2\theta^{\left(-\frac{1}{2}\right)+1}}{-\frac{1}{2}+1}\right]_1^4$$

$$= \left[\frac{\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2\theta^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4 = \left[\frac{2}{3}\sqrt{\theta^3} + 4\sqrt{\theta}\right]_1^4$$

$$= \left\{\frac{2}{3}\sqrt{(4)^3} + 4\sqrt{4}\right\} - \left\{\frac{2}{3}\sqrt{(1)^3} + 4\sqrt{1}\right\}$$

$$= \left\{\frac{16}{3} + 8\right\} - \left\{\frac{2}{3} + 4\right\}$$

$$= 5\frac{1}{3} + 8 - \frac{2}{3} - 4 = \mathbf{8\frac{2}{3}}$$

Problem 14. Evaluate: $\int_0^{\pi/2} 3\sin 2x\,dx$

$$\int_0^{\frac{\pi}{2}} 3\sin 2x\,dx$$

$$= \left[(3)\left(-\frac{1}{2}\right)\cos 2x\right]_0^{\frac{\pi}{2}} = \left[-\frac{3}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \left\{-\frac{3}{2}\cos 2\left(\frac{\pi}{2}\right)\right\} - \left\{-\frac{3}{2}\cos 2(0)\right\}$$

$$= \left\{-\frac{3}{2}\cos\pi\right\} - \left\{-\frac{3}{2}\cos 0\right\}$$

$$= \left\{-\frac{3}{2}(-1)\right\} - \left\{-\frac{3}{2}(1)\right\} = \frac{3}{2} + \frac{3}{2} = \mathbf{3}$$

Problem 15. Evaluate $\int_1^2 4\cos 3t\,dt$

$$\int_1^2 4\cos 3t\,dt = \left[(4)\left(\frac{1}{3}\right)\sin 3t\right]_1^2 = \left[\frac{4}{3}\sin 3t\right]_1^2$$

$$= \left\{\frac{4}{3}\sin 6\right\} - \left\{\frac{4}{3}\sin 3\right\}$$

Note that limits of trigonometric functions are always expressed in radians—thus, for example, sin 6 means the sine of 6 radians $= -0.279415\ldots$

Hence $$\int_1^2 4\cos 3t\,dt = \left\{\frac{4}{3}(-0.279415\ldots)\right\} - \left\{\frac{4}{3}(-0.141120\ldots)\right\}$$

$$= (-0.37255) - (0.18816) = \mathbf{-0.5607}$$

Problem 16. Evaluate

(a) $\int_1^2 4e^{2x}\,dx$ (b) $\int_1^4 \frac{3}{4u}du$,

each correct to 4 significant figures

(a) $$\int_1^2 4e^{2x}\,dx = \left[\frac{4}{2}e^{2x}\right]_1^2$$

$$= 2[e^{2x}]_1^2 = 2[e^4 - e^2]$$

$$= 2[54.5982 - 7.3891] = \mathbf{94.42}$$

(b) $$\int_1^4 \frac{3}{4u}du = \left[\frac{3}{4}\ln u\right]_1^4 = \frac{3}{4}[\ln 4 - \ln 1]$$

$$= \frac{3}{4}[1.3863 - 0] = \mathbf{1.040}$$

Now try the following exercise

Exercise 173 Further problems on definite integrals

In Problems 1 to 8, evaluate the definite integrals (where necessary, correct to 4 significant figures).

1. (a) $\int_1^4 5x^2 dx$ (b) $\int_{-1}^1 -\frac{3}{4}t^2 dt$
   $\left[\text{(a) } 105 \quad \text{(b) } -\frac{1}{2}\right]$

2. (a) $\int_{-1}^2 (3 - x^2)dx$ (b) $\int_1^3 (x^2 - 4x + 3)dx$
   $\left[\text{(a) } 6 \quad \text{(b) } -1\frac{1}{3}\right]$

3. (a) $\int_0^\pi \frac{3}{2}\cos\theta\,d\theta$ (b) $\int_0^{\frac{\pi}{2}} 4\cos\theta\,d\theta$
   [(a) 0 (b) 4]

4. (a) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sin 2\theta\,d\theta$ (b) $\int_0^2 3\sin t\,dt$
   [(a) 1 (b) 4.248]

5. (a) $\int_0^1 5\cos 3x\,dx$ (b) $\int_0^{\frac{\pi}{6}} 3\sec^2 2x\,dx$
   [(a) 0.2352 (b) 2.598]

6. (a) $\int_1^2 \text{cosec}^2 4t\,dt$

   (b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (3\sin 2x - 2\cos 3x)dx$
   [(a) 0.2572 (b) 2.638]

7. (a) $\int_0^1 3e^{3t}\,dt$ (b) $\int_{-1}^2 \frac{2}{3e^{2x}}dx$
   [(a) 19.09 (b) 2.457]

8. (a) $\int_2^3 \frac{2}{3x}dx$ (b) $\int_1^3 \frac{2x^2+1}{x}dx$
   [(a) 0.2703 (b) 9.099]

9. The entropy change ΔS, for an ideal gas is given by:

$$\Delta S = \int_{T_1}^{T_2} C_v \frac{dT}{T} - R\int_{V_1}^{V_2} \frac{dV}{V}$$

where T is the thermodynamic temperature, V is the volume and $R = 8.314$. Determine the entropy change when a gas expands from 1 litre to 3 litres for a temperature rise from 100 K to 400 K given that:

$$C_v = 45 + 6 \times 10^{-3}T + 8 \times 10^{-6}T^2$$

[55.65]

10. The p.d. between boundaries a and b of an electric field is given by: $V = \int_a^b \frac{Q}{2\pi r \varepsilon_0 \varepsilon_r} dr$
If $a = 10$, $b = 20$, $Q = 2 \times 10^{-6}$ coulombs, $\varepsilon_0 = 8.85 \times 10^{-12}$ and $\varepsilon_r = 2.77$, show that $V = 9$ kV.

11. The average value of a complex voltage waveform is given by:

$$V_{AV} = \frac{1}{\pi}\int_0^{\pi} (10 \sin \omega t + 3 \sin 3\omega t + 2 \sin 5\omega t) d(\omega t)$$

Evaluate V_{AV} correct to 2 decimal places.

[7.26]

Chapter 49

Integration using algebraic substitutions

49.1 Introduction

Functions that require integrating are not always in the 'standard form' shown in Chapter 48. However, it is often possible to change a function into a form which can be integrated by using either:

(i) an algebraic substitution (see Section 49.2),

(ii) trigonometric substitutions (see Chapter 50),

(iii) partial fractions (see Chapter 51),

(iv) the $t=\tan\frac{\theta}{2}$ substitution (see Chapter 52), or

(v) integration by parts (see Chapter 53).

49.2 Algebraic substitutions

With **algebraic substitutions**, the substitution usually made is to let u be equal to $f(x)$ such that $f(u)\,du$ is a standard integral. It is found that integrals of the forms:

$$k\int [f(x)]^n f'(x)dx \quad\text{and}\quad k\int \frac{f'(x)^n}{[f(x)]}dx$$

(where k and n are constants) can both be integrated by substituting u for $f(x)$.

49.3 Worked problems on integration using algebraic substitutions

Problem 1. Determine $\int \cos(3x+7)\,dx$

$\int \cos(3x+7)\,dx$ is not a standard integral of the form shown in Table 48.1, page 436, thus an algebraic substitution is made.

Let $u=3x+7$ then $\frac{du}{dx}=3$ and rearranging gives $dx=\frac{du}{3}$

Hence $\int \cos(3x+7)\,dx = \int (\cos u)\frac{du}{3}$

$$= \int \frac{1}{3}\cos u\,du,$$

which is a standard integral

$$= \frac{1}{3}\sin u + c$$

Rewriting u as $(3x+7)$ gives:

$$\int \cos(3x+7)\,dx = \mathbf{\frac{1}{3}\sin(3x+7)+c},$$

which may be checked by differentiating it.

Problem 2. Find: $\int (2x-5)^7 dx$

$(2x-5)$ may be multiplied by itself 7 times and then each term of the result integrated. However, this would be a lengthy process, and thus an algebraic substitution is made.

Let $u=(2x-5)$ then $\frac{du}{dx}=2$ and $dx=\frac{du}{2}$

Hence

$$\int (2x-5)^7 dx = \int u^7\frac{du}{2} = \frac{1}{2}\int u^7 du$$

$$= \frac{1}{2}\left(\frac{u^8}{8}\right)+c = \frac{1}{16}u^8+c$$

Rewriting u as $(2x-5)$ gives:

$$\int (2x-5)^7 dx = \frac{1}{16}(2x-5)^8 + c$$

Problem 3. Find: $\int \frac{4}{(5x-3)} dx$

Let $u=(5x-3)$ then $\frac{du}{dx}=5$ and $dx=\frac{du}{5}$

Hence $\int \frac{4}{(5x-3)}dx = \int \frac{4}{u}\frac{du}{5} = \frac{4}{5}\int \frac{1}{u}du$

$$= \frac{4}{5}\ln u + c$$

$$= \frac{4}{5}\ln(5x-3)+c$$

Problem 4. Evaluate $\int_0^1 2e^{6x-1}dx$, correct to 4 significant figures

Let $u=6x-1$ then $\frac{du}{dx}=6$ and $dx=\frac{du}{6}$

Hence $\int 2e^{6x-1}dx = \int 2e^u \frac{du}{6} = \frac{1}{3}\int e^u du$

$$= \frac{1}{3}e^u + c = \frac{1}{3}e^{6x-1}+c$$

Thus $\int_0^1 2e^{6x-1}dx = \frac{1}{3}\left[e^{6x-1}\right]_0^1$

$$= \frac{1}{3}[e^5 - e^{-1}] = 49.35,$$

correct to 4 significant figures.

Problem 5. Determine: $\int 3x(4x^2+3)^5 dx$

Let $u=(4x^2+3)$ then $\frac{du}{dx}=8x$ and $dx=\frac{du}{8x}$

Hence

$$\int 3x(4x^2+3)^5 dx = \int 3x(u)^5 \frac{du}{8x}$$

$$= \frac{3}{8}\int u^5 du, \text{ by cancelling}$$

The original variable 'x' has been completely removed and the integral is now only in terms of u and is a standard integral.

Hence $\frac{3}{8}\int u^5 du = \frac{3}{8}\left(\frac{u^6}{6}\right)+c = \frac{1}{16}u^6 + c$

$$= \frac{1}{16}(4x^2+3)^6 + c$$

Problem 6. Evaluate: $\int_0^{\pi/6} 24\sin^5\theta\cos\theta\, d\theta$

Let $u=\sin\theta$ then $\frac{du}{d\theta}=\cos\theta$ and $d\theta=\frac{du}{\cos\theta}$

Hence $\int 24\sin^5\theta\cos\theta\, d\theta$

$$= \int 24u^5\cos\theta\frac{du}{\cos\theta}$$

$$= 24\int u^5 du, \text{ by cancelling}$$

$$= 24\frac{u^6}{6}+c = 4u^6+c = 4(\sin\theta)^6+c$$

$$= 4\sin^6\theta + c$$

Thus $\int_0^{\pi/6} 24\sin^5\theta\cos\theta\, d\theta$

$$= \left[4\sin^6\theta\right]_0^{\pi/6} = 4\left[\left(\sin\frac{\pi}{6}\right)^6 - (\sin 0)^6\right]$$

$$= 4\left[\left(\frac{1}{2}\right)^6 - 0\right] = \frac{1}{16} \text{ or } 0.0625$$

Now try the following exercise

Exercise 174 Further problems on integration using algebraic substitutions

In Problems 1 to 6, integrate with respect to the variable.

1. $2\sin(4x+9)$ $\quad\left[-\frac{1}{2}\cos(4x+9)+c\right]$
2. $3\cos(2\theta-5)$ $\quad\left[\frac{3}{2}\sin(2\theta-5)+c\right]$

3. $4\sec^2(3t+1)$ $\left[\frac{4}{3}\tan(3t+1)+c\right]$

4. $\frac{1}{2}(5x-3)^6$ $\left[\frac{1}{70}(5x-3)^7+c\right]$

5. $\frac{-3}{(2x-1)}$ $\left[-\frac{3}{2}\ln(2x-1)+c\right]$

6. $3e^{3\theta+5}$ $[e^{3\theta+5}+c]$

In Problems 7 to 10, evaluate the definite integrals correct to 4 significant figures.

7. $\int_0^1 (3x+1)^5 dx$ [227.5]

8. $\int_0^2 x\sqrt{2x^2+1}\,dx$ [4.333]

9. $\int_0^{\pi/3} 2\sin\left(3t+\frac{\pi}{4}\right) dt$ [0.9428]

10. $\int_0^1 3\cos(4x-3)dx$ [0.7369]

49.4 Further worked problems on integration using algebraic substitutions

Problem 7. Find: $\int \frac{x}{2+3x^2}dx$

Let $u=2+3x^2$ then $\frac{du}{dx}=6x$ and $dx=\frac{du}{6x}$

Hence $\int \frac{x}{2+3x^2}dx$

$$= \int \frac{x}{u}\frac{du}{6x} = \frac{1}{6}\int \frac{1}{u}du, \text{ by cancelling,}$$

$$= \frac{1}{6}\ln u + x$$

$$= \mathbf{\frac{1}{6}\ln(2+3x^2)+c}$$

Problem 8. Determine: $\int \frac{2x}{\sqrt{4x^2-1}}dx$

Let $u=4x^2-1$ then $\frac{du}{dx}=8x$ and $dx=\frac{du}{8x}$

Hence $\int \frac{2x}{\sqrt{4x^2-1}}dx$

$$= \int \frac{2x}{\sqrt{u}}\frac{du}{8x} = \frac{1}{4}\int \frac{1}{\sqrt{u}}du, \text{ by cancelling}$$

$$= \frac{1}{4}\int u^{-1/2}du$$

$$= \frac{1}{4}\left[\frac{u^{(-1/2)+1}}{-\frac{1}{2}+1}\right]+c = \frac{1}{4}\left[\frac{u^{1/2}}{\frac{1}{2}}\right]+c$$

$$= \frac{1}{2}\sqrt{u}+c = \mathbf{\frac{1}{2}\sqrt{4x^2-1}+c}$$

Problem 9. Show that:

$$\int \tan\theta\, d\theta = \ln(\sec\theta)+c$$

$$\int \tan\theta\, d\theta = \int \frac{\sin\theta}{\cos\theta}d\theta.$$

Let u = $\cos\theta$

then $\frac{du}{d\theta}=-\sin\theta$ and $d\theta=\frac{-du}{\sin\theta}$

Hence

$$\int \frac{\sin\theta}{\cos\theta}d\theta = \int \frac{\sin\theta}{u}\left(\frac{-du}{\sin\theta}\right)$$

$$= -\int \frac{1}{u}du = -\ln u + c$$

$$= -\ln(\cos\theta)+c$$

$$= \ln(\cos\theta)^{-1}+c,$$

by the laws of logarithms

Hence $\int \mathbf{\tan\theta\, d\theta = \ln(\sec\theta)+c}$,

since $(\cos\theta)^{-1} = \frac{1}{\cos\theta} = \sec\theta$

49.5 Change of limits

When evaluating definite integrals involving substitutions it is sometimes more convenient to **change the limits** of the integral as shown in Problems 10 and 11.

Problem 10. Evaluate: $\int_1^3 5x\sqrt{2x^2+7}\,dx$, taking positive values of square roots only

Let $u = 2x^2 + 7$, then $\frac{du}{dx} = 4x$ and $dx = \frac{du}{4x}$

It is possible in this case to change the limits of integration. Thus when $x = 3$, $u = 2(3)^2 + 7 = 25$ and when $x = 1$, $u = 2(1)^2 + 7 = 9$

$$\text{Hence } \int_{x=1}^{x=3} 5x\sqrt{2x^2+7}\,dx = \int_{u=9}^{u=25} 5x\sqrt{u}\,\frac{du}{4x} = \frac{5}{4}\int_9^{25}\sqrt{u}\,du = \frac{5}{4}\int_9^{25} u^{1/2}du$$

Thus the limits have been changed, and it is unnecessary to change the integral back in terms of x.

$$\text{Thus } \int_{x=1}^{x=3} 5x\sqrt{2x^2+7}\,dx = \frac{5}{4}\left[\frac{u^{3/2}}{3/2}\right]_9^{25} = \frac{5}{6}\left[\sqrt{u^3}\right]_9^{25} = \frac{5}{6}[\sqrt{25^3} - \sqrt{9^3}] = \frac{5}{6}(125 - 27) = \mathbf{81\frac{2}{3}}$$

Problem 11. Evaluate: $\int_0^2 \frac{3x}{\sqrt{2x^2+1}}\,dx$, taking positive values of square roots only

Let $u = 2x^2 + 1$ then $\frac{du}{dx} = 4x$ and $dx = \frac{du}{4x}$

$$\text{Hence } \int_0^2 \frac{3x}{\sqrt{2x^2+1}}\,dx = \int_{x=0}^{x=2} \frac{3x}{\sqrt{u}}\,\frac{du}{4x} = \frac{3}{4}\int_{x=0}^{x=2} u^{-1/2}du$$

Since $u = 2x^2 + 1$, when $x = 2$, $u = 9$ and when $x = 0$, $u = 1$

Thus $\frac{3}{4}\int_{x=0}^{x=2} u^{-1/2}\,du = \frac{3}{4}\int_{u=1}^{u=9} u^{-1/2}\,du$,

i.e. the limits have been changed

$$= \frac{3}{4}\left[\frac{u^{1/2}}{\frac{1}{2}}\right]_1^9 = \frac{3}{2}[\sqrt{9} - \sqrt{1}] = \mathbf{3},$$

taking positive values of square roots only.

Now try the following exercise

Exercise 175 Further problems on integration using algebraic substitutions

In Problems 1 to 7, integrate with respect to the variable.

1. $2x(2x^2-3)^5$ $\quad\left[\frac{1}{12}(2x^2-3)^6 + c\right]$
2. $5\cos^5 t\sin t$ $\quad\left[-\frac{5}{6}\cos^6 t + c\right]$
3. $3\sec^2 3x\tan 3x$ $\quad\left[\frac{1}{2}\sec^2 3x + c \text{ or } \frac{1}{2}\tan^2 3x + c\right]$
4. $2t\sqrt{3t^2-1}$ $\quad\left[\frac{2}{9}\sqrt{(3t^2-1)^3} + c\right]$
5. $\frac{\ln\theta}{\theta}$ $\quad\left[\frac{1}{2}(\ln\theta)^2 + c\right]$
6. 3 tan 2t $\quad\left[\frac{3}{2}\ln(\sec 2t) + c\right]$
7. $\frac{2e^t}{\sqrt{e^t+4}}$ $\quad[4\sqrt{e^t+4} + c]$

In Problems 8 to 10, evaluate the definite integrals correct to 4 significant figures.

8. $\int_0^1 3xe^{(2x^2-1)}dx$ $\quad$ [1.763]
9. $\int_0^{\pi/2} 3\sin^4\theta\cos\theta\,d\theta$ $\quad$ [0.6000]
10. $\int_0^1 \frac{3x}{(4x^2-1)^5}dx$ $\quad$ [0.09259]

11. The electrostatic potential on all parts of a conducting circular disc of radius r is given by the equation:

$$V = 2\pi\sigma \int_0^9 \frac{R}{\sqrt{R^2 + r^2}} dR$$

Solve the equation by determining the integral. $\left[V = 2\pi\sigma\left\{\sqrt{(9^2 + r^2)} - r\right\}\right]$

12. In the study of a rigid rotor the following integration occurs:

$$Z_r = \int_0^\infty (2J+1)\, e^{\frac{-J(J+1)\,h^2}{8\pi^2 IkT}}\, dJ$$

Determine Z_r for constant temperature T assuming h, I and k are constants. $\left[\frac{8\pi^2 IkT}{h^2}\right]$

13. In electrostatics,

$$E = \int_0^\pi \left\{ \frac{a^2\sigma \sin\theta}{2\,\varepsilon\sqrt{(a^2 - x^2 - 2ax\cos\theta)}} d\theta \right\}$$

where a, σ and ε are constants, x is greater than a, and x is independent of θ. Show that $E = \frac{a^2\sigma}{\varepsilon x}$

Chapter 50

Integration using trigonometric substitutions

50.1 Introduction

Table 50.1 gives a summary of the integrals that require the use of **trigonometric substitutions**, and their application is demonstrated in Problems 1 to 19.

50.2 Worked problems on integration of $\sin^2 x$, $\cos^2 x$, $\tan^2 x$ and $\cot^2 x$

Problem 1. Evaluate: $\int_0^{\frac{\pi}{4}} 2\cos^2 4t\, dt$

Since $\quad \cos 2t = 2\cos^2 t - 1$ (from Chapter 27),

then $\quad \cos^2 t = \frac{1}{2}(1 + \cos 2t)$ and

$$\cos^2 4t = \frac{1}{2}(1 + \cos 8t)$$

Hence $\int_0^{\frac{\pi}{4}} 2\cos^2 4t\, dt$

$$= 2\int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 8t)\, dt$$

$$= \left[t + \frac{\sin 8t}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{4} + \frac{\sin 8\left(\frac{\pi}{4}\right)}{8}\right] - \left[0 + \frac{\sin 0}{8}\right]$$

$$= \boldsymbol{\frac{\pi}{4}} \text{ or } \mathbf{0.7854}$$

Problem 2. Determine: $\int \sin^2 3x\, dx$

Since $\quad \cos 2x = 1 - 2\sin^2 x$ (from Chapter 27),

then $\quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and

$$\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$$

Hence $\int \sin^2 3x\, dx = \int \frac{1}{2}(1 - \cos 6x)\, dx$

$$= \boldsymbol{\frac{1}{2}\left(x - \frac{\sin 6x}{6}\right) + c}$$

Problem 3. Find: $3\int \tan^2 4x\, dx$

Since $1 + \tan^2 x = \sec^2 x$, then $\tan^2 x = \sec^2 x - 1$ and $\tan^2 4x = \sec^2 4x - 1$

Hence $\quad 3\int \tan^2 4x\, dx = 3\int (\sec^2 4x - 1)\, dx$

$$= \boldsymbol{3\left(\frac{\tan 4x}{4} - x\right) + c}$$

Problem 4. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}\cot^2 2\theta\, d\theta$

Table 50.1 Integrals using trigonometric substitutions

$f(x)$	$\int f(x)dx$	Method	See problem
1. $\cos^2 x$	$\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)+c$	Use $\cos 2x = 2\cos^2 x - 1$	1
2. $\sin^2 x$	$\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)+c$	Use $\cos 2x = 1 - 2\sin^2 x$	2
3. $\tan^2 x$	$\tan x - x + c$	Use $1+\tan^2 x = \sec^2 x$	3
4. $\cot^2 x$	$-\cot x - x + c$	Use $\cot^2 x + 1 = \text{cosec}^2 x$	4
5. $\cos^m x \sin^n x$	(a) If either m or n is odd (but not both), use $\cos^2 x + \sin^2 x = 1$		5, 6
	(b) If both m and n are even, use either $\cos 2x = 2\cos^2 x - 1$ or $\cos 2x = 1 - 2\sin^2 x$		7, 8
6. $\sin A \cos B$		Use $\frac{1}{2}[\sin(A+B)+\sin(A-B)]$	9
7. $\cos A \sin B$		Use $\frac{1}{2}[\sin(A+B)-\sin(A-B)]$	10
8. $\cos A \cos B$		Use $\frac{1}{2}[\cos(A+B)+\cos(A-B)]$	11
9. $\sin A \sin B$		Use $-\frac{1}{2}[\cos(A+B)-\cos(A-B)]$	12
10. $\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}+c$	Use $x = a\sin\theta$ substitution	13, 14
11. $\sqrt{a^2-x^2}$	$\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c$	Use $x = a\sin\theta$ substitution	15, 16
12. $\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}+c$	Use $x = a\tan\theta$ substitution	17–19

Since $\cot^2\theta + 1 = \text{cosec}^2\theta$, then

$\cot^2\theta = \text{cosec}^2\theta - 1$ and $\cot^2 2\theta = \text{cosec}^2 2\theta - 1$

Hence $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}\cot^2 2\theta\, d\theta$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\text{cosec}^2 2\theta - 1)\, d\theta = \frac{1}{2}\left[\frac{-\cot 2\theta}{2} - \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\left(\frac{-\cot 2\left(\frac{\pi}{3}\right)}{2} - \frac{\pi}{3}\right) - \left(\frac{-\cot 2\left(\frac{\pi}{6}\right)}{2} - \frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}[(--0.2887 - 1.0472) - (-0.2887 - 0.5236)]$$

$$= \mathbf{0.0269}$$

Now try the following exercise

Exercise 176 Further problems on integration of $\sin^2 x$, $\cos^2 x$, $\tan^2 x$ and $\cot^2 x$

In Problems 1 to 4, integrate with respect to the variable.

1. $\sin^2 2x$ $\left[\frac{1}{2}\left(x - \frac{\sin 4x}{4}\right) + c\right]$
2. $3\cos^2 t$ $\left[\frac{3}{2}\left(t + \frac{\sin 2t}{2}\right) + c\right]$
3. $5\tan^2 3\theta$ $\left[5\left(\frac{1}{3}\tan 3\theta - \theta\right) + c\right]$
4. $2\cot^2 2t$ $[-(\cot 2t + 2t) + c]$

In Problems 5 to 8, evaluate the definite integrals, correct to 4 significant figures.

5. $\int_0^{\pi/3} 3\sin^2 3x\, dx$ $\left[\frac{\pi}{2} \text{ or } 1.571\right]$
6. $\int_0^{\pi/4} \cos^2 4x\, dx$ $\left[\frac{\pi}{8} \text{ or } 0.3927\right]$
7. $\int_0^{0.5} 2\tan^2 2t\, dt$ $[0.5574]$
8. $\int_{\pi/6}^{\pi/3} \cot^2 \theta\, d\theta$ $[0.6311]$

50.3 Worked problems on powers of sines and cosines

Problem 5. Determine: $\int \sin^5 \theta\, d\theta$

Since $\cos^2\theta + \sin^2\theta = 1$ then $\sin^2\theta = (1 - \cos^2\theta)$.

Hence $\int \sin^5\theta\, d\theta$

$$= \int \sin\theta(\sin^2\theta)^2\, d\theta = \int \sin\theta(1-\cos^2\theta)^2 d\theta$$

$$= \int \sin\theta(1 - 2\cos^2\theta + \cos^4\theta)\, d\theta$$

$$= \int (\sin\theta - 2\sin\theta\cos^2\theta + \sin\theta\cos^4\theta) d\theta$$

$$= \mathbf{-\cos\theta + \frac{2\cos^3\theta}{3} - \frac{\cos^5\theta}{5} + c}$$

[Whenever a power of a cosine is multiplied by a sine of power 1, or vice-versa, the integral may be determined by inspection as shown.

In general, $\int \cos^n\theta \sin\theta\, d\theta = \frac{-\cos^{n+1}\theta}{(n+1)} + c$

and $\int \sin^n\theta\cos\theta\, d\theta = \frac{\sin^{n+1}\theta}{(n+1)} + c$

Alternatively, an algebraic substitution may be used as shown in Problem 6, chapter 49, page 443].

Problem 6. Evaluate: $\int_0^{\frac{\pi}{2}} \sin^2 x\cos^3 x\, dx$

$$\int_0^{\frac{\pi}{2}} \sin^2 x\cos^3 x\, dx = \int_0^{\frac{\pi}{2}} \sin^2 x\cos^2 x\cos x\, dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x)(1 - \sin^2 x)(\cos x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x\cos x - \sin^4 x\cos x)\, dx$$

$$= \left[\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\left(\sin\frac{\pi}{2}\right)^3}{3} - \frac{\left(\sin\frac{\pi}{2}\right)^5}{5}\right] - [0 - 0]$$

$$= \frac{1}{3} - \frac{1}{5} = \mathbf{\frac{2}{15}} \text{ or } \mathbf{0.1333}$$

Problem 7. Evaluate: $\int_0^{\frac{\pi}{4}} 4\cos^4\theta\, d\theta$, correct to 4 significant figures

$$\int_0^{\frac{\pi}{4}} 4\cos^4\theta\, d\theta = 4\int_0^{\frac{\pi}{4}} (\cos^2\theta)^2 d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \left[\frac{1}{2}(1 + \cos 2\theta)\right]^2 d\theta$$

$$= \int_0^{\frac{\pi}{4}} (1 + 2\cos 2\theta + \cos^2 2\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[1 + 2\cos 2\theta + \frac{1}{2}(1 + \cos 4\theta)\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right] d\theta$$

$$= \left[\frac{3\theta}{2} + \sin 2\theta + \frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{3}{2}\left(\frac{\pi}{4}\right) + \sin\frac{2\pi}{4} + \frac{\sin 4(\pi/4)}{8}\right] - [0]$$

$$= \frac{3\pi}{8} + 1$$

$= \mathbf{2.178}$, correct to 4 significant figures.

Problem 8. Find: $\int \sin^2 t \cos^4 t\, dt$

$$\int \sin^2 t \cos^4 t\, dt = \int \sin^2 t(\cos^2 t)^2\, dt$$

$$= \int \left(\frac{1-\cos 2t}{2}\right)\left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$$= \frac{1}{8}\int (1-\cos 2t)(1+2\cos 2t+\cos^2 2t)\, dt$$

$$= \frac{1}{8}\int (1+2\cos 2t+\cos^2 2t-\cos 2t - 2\cos^2 2t - \cos^3 2t)\, dt$$

$$= \frac{1}{8}\int (1+\cos 2t-\cos^2 2t-\cos^3 2t)\, dt$$

$$= \frac{1}{8}\int \left[1+\cos 2t - \left(\frac{1+\cos 4t}{2}\right) - \cos 2t(1-\sin^2 2t)\right] dt$$

$$= \frac{1}{8}\int \left(\frac{1}{2} - \frac{\cos 4t}{2} + \cos 2t \sin^2 2t\right) dt$$

$$= \mathbf{\frac{1}{8}\left(\frac{t}{2} - \frac{\sin 4t}{8} + \frac{\sin^3 2t}{6}\right) + c}$$

Now try the following exercise

Exercise 177 Further problems on integration of powers of sines and cosines

Integrate the following with respect to the variable:

1. $\sin^3\theta$ $\quad \left[-\cos\theta + \dfrac{\cos^3\theta}{3} + c\right]$
2. $2\cos^3 2x$ $\quad \left[\sin 2x - \dfrac{\sin^3 2x}{3} + c\right]$
3. $2\sin^3 t\cos^2 t$ $\quad \left[\dfrac{-2}{3}\cos^3 t + \dfrac{2}{5}\cos^5 t + c\right]$
4. $\sin^3 x\cos^4 x$ $\quad \left[\dfrac{-\cos^5 x}{5} + \dfrac{\cos^7 x}{7} + c\right]$
5. $2\sin^4 2\theta$ $\quad \left[\dfrac{3\theta}{4} - \dfrac{1}{4}\sin 4\theta + \dfrac{1}{32}\sin 8\theta + c\right]$
6. $\sin^2 t\cos^2 t$ $\quad \left[\dfrac{t}{8} - \dfrac{1}{32}\sin 4t + c\right]$

50.4 Worked problems on integration of products of sines and cosines

Problem 9. Determine: $\int \sin 3t \cos 2t\, dt$

$$\int \sin 3t\cos 2t\, dt = \int \frac{1}{2}[\sin(3t+2t)+\sin(3t-2t)]\, dt,$$

from 6 of Table 50.1, which follows from Section 27.4, page 238,

$$= \frac{1}{2}\int (\sin 5t + \sin t)\, dt$$

$$= \mathbf{\frac{1}{2}\left(\frac{-\cos 5t}{5} - \cos t\right) + c}$$

Problem 10. Find: $\int \frac{1}{3}\cos 5x \sin 2x\, dx$

$$\int \frac{1}{3}\cos 5x\sin 2x\, dx = \frac{1}{3}\int \frac{1}{2}[\sin(5x+2x) - \sin(5x-2x)]\, dx,$$

from 7 of Table 50.1

$$= \frac{1}{6}\int (\sin 7x - \sin 3x)\, dx$$

$$= \mathbf{\frac{1}{6}\left(\frac{-\cos 7x}{7} + \frac{\cos 3x}{3}\right) + c}$$

Problem 11. Evaluate: $\int_0^1 2\cos 6\theta \cos\theta\, d\theta$, correct to 4 decimal places

$$\int_0^1 2\cos 6\theta\cos\theta\, d\theta$$

$$= 2\int_0^1 \frac{1}{2}[\cos(6\theta+\theta)+\cos(6\theta-\theta)]\, d\theta,$$

from 8 of Table 50.1

$$= \int_0^1 (\cos 7\theta + \cos 5\theta)\, d\theta = \left[\frac{\sin 7\theta}{7}+\frac{\sin 5\theta}{5}\right]_0^1$$

$$= \left(\frac{\sin 7}{7}+\frac{\sin 5}{5}\right) - \left(\frac{\sin 0}{7}+\frac{\sin 0}{5}\right)$$

'sin 7' means 'the sine of 7 radians' ($\equiv 401.07°$) and $\sin 5 \equiv 286.48°$.

Hence $\int_0^1 2\cos 6\theta\cos\theta\, d\theta$

$= (0.09386 + -0.19178) - (0)$

$= \mathbf{-0.0979}$, correct to 4 decimal places

Problem 12. Find: $3\int \sin 5x \sin 3x\, dx$

$$3\int \sin 5x \sin 3x\, dx$$

$$= 3\int -\frac{1}{2}[\cos(5x+3x)-\cos(5x-3x)]\, dx,$$

from 9 of Table 50.1

$$= -\frac{3}{2}\int (\cos 8x - \cos 2x)\, dx$$

$$= \boldsymbol{-\frac{3}{2}\left(\frac{\sin 8x}{8}-\frac{\sin 2x}{2}\right)+c} \quad \text{or} \quad \boldsymbol{\frac{3}{16}(4\sin 2x - \sin 8x)+c}$$

Now try the following exercise

Exercise 178 Further problems on integration of products of sines and cosines

In Problems 1 to 4, integrate with respect to the variable.

1. $\sin 5t \cos 2t$ $\quad\left[-\frac{1}{2}\left(\frac{\cos 7t}{7}+\frac{\cos 3t}{3}\right)+c\right]$

2. $2\sin 3x \sin x$ $\quad\left[\frac{\sin 2x}{2}-\frac{\sin 4x}{4}+c\right]$

3. $3\cos 6x\cos x$ $\quad\left[\frac{3}{2}\left(\frac{\sin 7x}{7}+\frac{\sin 5x}{5}\right)+c\right]$

4. $\frac{1}{2}\cos 4\theta \sin 2\theta$ $\quad\left[\frac{1}{4}\left(\frac{\cos 2\theta}{2}-\frac{\cos 6\theta}{6}\right)+c\right]$

In Problems 5 to 8, evaluate the definite integrals.

5. $\int_0^{\pi/2} \cos 4x \cos 3x\, dx$ $\quad\left[\frac{3}{7} \text{ or } 0.4286\right]$

6. $\int_0^1 2\sin 7t\cos 3t\, dt$ $\quad[0.5973]$

7. $-4\int_0^{\pi/3} \sin 5\theta \sin 2\theta\, d\theta$ $\quad[0.2474]$

8. $\int_1^2 3\cos 8t \sin 3t\, dt$ $\quad[-0.1999]$

50.5 Worked problems on integration using the sin θ substitution

Problem 13. Determine: $\int \frac{1}{\sqrt{a^2-x^2}}\, dx$

Let $x = a\sin\theta$, then $\frac{dx}{d\theta} = a\cos\theta$ and $dx = a\cos\theta\, d\theta$.

Hence $\int \frac{1}{\sqrt{a^2-x^2}}\, dx$

$$= \int \frac{1}{\sqrt{a^2-a^2\sin^2\theta}}\, a\cos\theta\, d\theta$$

$$= \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2(1-\sin^2\theta)}}$$

$$= \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2\cos^2\theta}}, \text{ since } \sin^2\theta + \cos^2\theta = 1$$

$$= \int \frac{a\cos\theta\, d\theta}{a\cos\theta} = \int d\theta = \theta + c$$

Since $x = a\sin\theta$, then $\sin\theta = \frac{x}{a}$ and $\theta = \sin^{-1}\frac{x}{a}$

Hence $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \boldsymbol{\sin^{-1}\frac{x}{a} + c}$

Problem 14. Evaluate $\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$

From Problem 13, $\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$

$$= \left[\sin^{-1}\frac{x}{3}\right]_0^3 \quad \text{since } a = 3$$

$$= (\sin^{-1}1 - \sin^{-1}0) = \frac{\pi}{2} \text{ or } \mathbf{1.5708}$$

Problem 15. Find: $\int \sqrt{a^2-x^2}\,dx$

Let $x = a\sin\theta$ then $\frac{dx}{d\theta} = a\cos\theta$ and $dx = a\cos\theta\,d\theta$

Hence $\int \sqrt{a^2-x^2}dx$

$$= \int \sqrt{a^2 - a^2\sin^2\theta}\,(a\cos\theta\,d\theta)$$

$$= \int \sqrt{a^2(1-\sin^2\theta)}\,(a\cos\theta\,d\theta)$$

$$= \int \sqrt{a^2\cos^2\theta}\,(a\cos\theta\,d\theta)$$

$$= \int (a\cos\theta)\,(a\cos\theta\,d\theta)$$

$$= a^2\int \cos^2\theta\,d\theta = a^2\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

(since $\cos 2\theta = 2\cos^2\theta - 1$)

$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{a^2}{2}\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right) + c$$

since from Chapter 27, $\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{a^2}{2}[\theta + \sin\theta\cos\theta] + c$$

Since $x = a\sin\theta$, then $\sin\theta = \frac{x}{a}$ and $\theta = \sin^{-1}\frac{x}{a}$

Also, $\cos^2\theta + \sin^2\theta = 1$, from which,

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{x}{a}\right)^2}$$

$$= \sqrt{\frac{a^2-x^2}{a^2}} = \frac{\sqrt{a^2-x^2}}{a}$$

Thus $\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}[\theta + \sin\theta\cos\theta]$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \left(\frac{x}{a}\right)\frac{\sqrt{a^2-x^2}}{a}\right] + c$$

$$= \boldsymbol{\frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c}$$

Problem 16. Evaluate: $\int_0^4 \sqrt{16-x^2}\,dx$

From Problem 15, $\int_0^4 \sqrt{16-x^2}\,dx$

$$= \left[\frac{16}{2}\sin^{-1}\frac{x}{4} + \frac{x}{2}\sqrt{16-x^2}\right]_0^4$$

$$= \left[8\sin^{-1}1 + 2\sqrt{0}\right] - \left[8\sin^{-1}0 + 0\right]$$

$$= 8\sin^{-1}1 = 8\left(\frac{\pi}{2}\right)$$

$$= \mathbf{4\pi} \text{ or } \mathbf{12.57}$$

Now try the following exercise

Exercise 179 Further problems on integration using the sine θ substitution

1. Determine: $\int \frac{5}{\sqrt{4-t^2}}\,dt$ $\left[5\sin^{-1}\frac{t}{2} + c\right]$

2. Determine: $\int \frac{3}{\sqrt{9-x^2}}\,dx$ $\left[3\sin^{-1}\frac{x}{3} + c\right]$

3. Determine: $\int \sqrt{4-x^2}\,dx$ $\left[2\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2} + c\right]$

4. Determine: $\int \sqrt{16-9t^2}\,dt$ $\left[\frac{8}{3}\sin^{-1}\frac{3t}{4} + \frac{t}{2}\sqrt{16-9t^2} + c\right]$

5. Evaluate: $\displaystyle\int_0^4 \frac{1}{\sqrt{16-x^2}}dx$ $\qquad \left[\frac{\pi}{2} \text{ or } 1.571\right]$

6. Evaluate: $\displaystyle\int_0^1 \sqrt{9-4x^2}\,dx$ $\qquad [2.760]$

50.6 Worked problems on integration using the tan θ substitution

Problem 17. Determine: $\displaystyle\int \frac{1}{(a^2+x^2)}dx$

Let $x = a\tan\theta$ then $\dfrac{dx}{d\theta} = a\sec^2\theta$ and $dx = a\sec^2\theta\,d\theta$

Hence $\displaystyle\int \frac{1}{(a^2+x^2)}dx$

$$= \int \frac{1}{(a^2+a^2\tan^2\theta)}(a\sec^2\theta\,d\theta)$$

$$= \int \frac{a\sec^2\theta\,d\theta}{a^2(1+\tan^2\theta)}$$

$$= \int \frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta} \text{ since } 1+\tan^2\theta = \sec^2\theta$$

$$= \int \frac{1}{a}d\theta = \frac{1}{a}(\theta)+c$$

Since $x = a\tan\theta$, $\theta = \tan^{-1}\dfrac{x}{a}$

Hence $\displaystyle\int \frac{1}{(a^2+x^2)}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

Problem 18. Evaluate: $\displaystyle\int_0^2 \frac{1}{(4+x^2)}dx$

From Problem 17, $\displaystyle\int_0^2 \frac{1}{(4+x^2)}dx$

$$= \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2 \quad \text{since } a = 2$$

$$= \frac{1}{2}(\tan^{-1}1 - \tan^{-1}0) = \frac{1}{2}\left(\frac{\pi}{4}-0\right)$$

$$= \frac{\pi}{8} \text{ or } \mathbf{0.3927}$$

Problem 19. Evaluate: $\displaystyle\int_0^1 \frac{5}{(3+2x^2)}dx$, correct to 4 decimal places

$$\int_0^1 \frac{5}{(3+2x^2)}dx = \int_0^1 \frac{5}{2[(3/2)+x^2]}dx$$

$$= \frac{5}{2}\int_0^1 \frac{1}{[\sqrt{3/2}]^2+x^2}dx$$

$$= \frac{5}{2}\left[\frac{1}{\sqrt{3/2}}\tan^{-1}\frac{x}{\sqrt{3/2}}\right]_0^1$$

$$= \frac{5}{2}\sqrt{\frac{2}{3}}\left[\tan^{-1}\sqrt{\frac{2}{3}} - \tan^{-1}0\right]$$

$$= (2.0412)[0.6847-0]$$

$= \mathbf{1.3976}$, correct to 4 decimal places.

Now try the following exercise

Exercise 180 Further problems on integration using the tan θ substitution

1. Determine: $\displaystyle\int \frac{3}{4+t^2}dt$ $\qquad \left[\frac{3}{2}\tan^{-1}\frac{x}{2}+c\right]$

2. Determine: $\displaystyle\int \frac{5}{16+9\theta^2}d\theta$ $\qquad \left[\frac{5}{12}\tan^{-1}\frac{3\theta}{4}+c\right]$

3. Evaluate: $\displaystyle\int_0^1 \frac{3}{1+t^2}dt$ $\qquad [2.356]$

4. Evaluate: $\displaystyle\int_0^3 \frac{5}{4+x^2}dx$ $\qquad [2.457]$

Revision Test 14

This Revision test covers the material contained in Chapters 48 to 50. *The marks for each question are shown in brackets at the end of each question.*

1. Determine:

   (a) $\int 3\sqrt{t^5}\,dt$

   (b) $\int \frac{2}{\sqrt[3]{x^2}}\,dx$

   (c) $\int (2+\theta)^2 d\theta$ (9)

2. Evaluate the following integrals, each correct to 4 significant figures:

   (a) $\int_0^{\pi/3} 3\sin 2t\,dt$

   (b) $\int_1^2 \left(\frac{2}{x^2}+\frac{1}{x}+\frac{3}{4}\right)dx$ (10)

3. Determine the following integrals:

   (a) $\int 5(6t+5)^7 dt$

   (b) $\int \frac{3\ln x}{x}dx$

   (c) $\int \frac{2}{\sqrt{(2\theta-1)}}\,d\theta$ (9)

4. Evaluate the following definite integrals:

   (a) $\int_0^{\pi/2} 2\sin\left(2t+\frac{\pi}{3}\right)dt$

   (b) $\int_0^1 3xe^{4x^2-3}\,dx$ (10)

5. Determine the following integrals:

   (a) $\int \cos^3 x \sin^2 x\,dx$

   (b) $\int \frac{2}{\sqrt{9-4x^2}}\,dx$ (8)

6. Evaluate the following definite integrals, correct to 4 significant figures:

   (a) $\int_0^{\pi/2} 3\sin^2 t\,dt$

   (b) $\int_0^{\pi/3} 3\cos 5\theta \sin 3\theta\,d\theta$

   (c) $\int_0^2 \frac{5}{4+x^2}\,dx$ (14)

Chapter 51

Integration using partial fractions

51.1 Introduction

The process of expressing a fraction in terms of simpler fractions—called **partial fractions**—is discussed in Chapter 7, with the forms of partial fractions used being summarised in Table 7.1, page 54.

Certain functions have to be resolved into partial fractions before they an be integrated, as demonstrated in the following worked problems.

51.2 Worked problems on integration using partial fractions with linear factors

Problem 1. Determine: $\int \frac{11-3x}{x^2+2x-3}\,dx$

As shown in Problem 1, page 54:

$$\frac{11-3x}{x^2+2x-3} \equiv \frac{2}{(x-1)} - \frac{5}{(x+3)}$$

Hence $\int \frac{11-3x}{x^2+2x-3}\,dx$

$$= \int \left\{\frac{2}{(x-1)} - \frac{5}{(x+3)}\right\} dx$$

$$= \mathbf{2\ln(x-1) - 5\ln(x+3) + c}$$

(by algebraic substitutions—see chapter 49)

or $\ln\left\{\frac{(x-1)^2}{(x+3)^5}\right\} + c$ by the laws of logarithms

Problem 2. Find: $\int \frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}\,dx$

It was shown in Problem 2, page 55:

$$\frac{2x^2-9x-35}{(x+1)(x-2)(x+3)} \equiv \frac{4}{(x+1)} - \frac{3}{(x-2)} + \frac{1}{(x+3)}$$

Hence $\int \frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}\,dx$

$$\equiv \int \left\{\frac{4}{(x+1)} - \frac{3}{(x-2)} + \frac{1}{(x+3)}\right\} dx$$

$$= \mathbf{4\ln(x+1) - 3\ln(x-2) + \ln(x+3) + c}$$

or $\ln\left\{\frac{(x+1)^4(x+3)}{(x-2)^3}\right\} + c$

Problem 3. Determine: $\int \frac{x^2+1}{x^2-3x+2}\,dx$

By dividing out (since the numerator and denominator are of the same degree) and resolving into partial fractions it was shown in Problem 3, page 55:

$$\frac{x^2+1}{x^2-3x+2} \equiv 1 - \frac{2}{(x-1)} + \frac{5}{(x-2)}$$

Hence $\int \frac{x^2+1}{x^2-3x+2}\,dx$

$$\equiv \int \left\{1 - \frac{2}{(x-1)} + \frac{5}{(x-2)}\right\} dx$$

$$= \mathbf{x - 2\ln(x-1) + 5\ln(x-2) + c}$$

$$\text{or } \mathbf{x + \ln\left\{\frac{(x-2)^5}{(x-1)^2}\right\} + c}$$

Problem 4. Evaluate:
$\displaystyle\int_2^3 \frac{x^3 - 2x^2 - 4x - 4}{x^2 + x - 2}\,dx$, correct to 4 significant figures

By dividing out and resolving into partial fractions, it was shown in Problem 4, page 56:

$$\frac{x^3 - 2x^2 - 4x - 4}{x^2 + x - 2} \equiv x - 3 + \frac{4}{(x+2)} - \frac{3}{(x-1)}$$

Hence $\displaystyle\int_2^3 \frac{x^3 - 2x^2 - 4x - 4}{x^2 + x - 2}\,dx$

$$\equiv \int_2^3 \left\{x - 3 + \frac{4}{(x+2)} - \frac{3}{(x-1)}\right\} dx$$

$$= \left[\frac{x^2}{2} - 3x + 4\ln(x+2) - 3\ln(x-1)\right]_2^3$$

$$= \left(\frac{9}{2} - 9 + 4\ln 5 - 3\ln 2\right) - (2 - 6 + 4\ln 4 - 3\ln 1)$$

$$= \mathbf{-1.687}, \text{ correct to 4 significant figures}$$

Now try the following exercise

Exercise 181 Further problems on integration using partial fractions with linear factors

In Problems 1 to 5, integrate with respect to x

1. $\displaystyle\int \frac{12}{(x^2-9)}\,dx$

$$\left[\begin{array}{c} 2\ln(x-3) - 2\ln(x+3) + c \\ \text{or } \ln\left\{\dfrac{x-3}{x+3}\right\}^2 + c \end{array}\right]$$

2. $\displaystyle\int \frac{4(x-4)}{(x^2-2x-3)}\,dx$

$$\left[\begin{array}{c} 5\ln(x+1) - \ln(x-3) + c \\ \text{or } \ln\left\{\dfrac{(x+1)^5}{(x-3)}\right\} + c \end{array}\right]$$

3. $\displaystyle\int \frac{3(2x^2-8x-1)}{(x+4)(x+1)(2x-1)}\,dx$

$$\left[\begin{array}{c} 7\ln(x+4) - 3\ln(x+1) - \ln(2x-1) + c \\ \text{or } \ln\left\{\dfrac{(x+4)^7}{(x+1)^3(2x-1)}\right\} + c \end{array}\right]$$

4. $\displaystyle\int \frac{x^2+9x+8}{x^2+x-6}\,dx$

$$\left[\begin{array}{c} x + 2\ln(x+3) + 6\ln(x-2) + c \\ \text{or } x + \ln\{(x+3)^2(x-2)^6\} + c \end{array}\right]$$

5. $\displaystyle\int \frac{3x^3 - 2x^2 - 16x + 20}{(x-2)(x+2)}\,dx$

$$\left[\begin{array}{c} \dfrac{3x^2}{2} - 2x + \ln(x-2) \\ -5\ln(x+2) + c \end{array}\right]$$

In Problems 6 and 7, evaluate the definite integrals correct to 4 significant figures.

6. $\displaystyle\int_3^4 \frac{x^2-3x+6}{x(x-2)(x-1)}\,dx$ [0.6275]

7. $\displaystyle\int_4^6 \frac{x^2-x-14}{x^2-2x-3}\,dx$ [0.8122]

51.3 Worked problems on integration using partial fractions with repeated linear factors

Problem 5. Determine: $\displaystyle\int \frac{2x+3}{(x-2)^2}\,dx$

It was shown in Problem 5, page 57:

$$\frac{2x+3}{(x-2)^2} \equiv \frac{2}{(x-2)} + \frac{7}{(x-2)^2}$$

Thus $\displaystyle\int \frac{2x+3}{(x-2)^2}\,dx$

$$\equiv \int \left\{\frac{2}{(x-2)} + \frac{7}{(x-2)^2}\right\} dx$$

$$= \mathbf{2\ln(x-2) - \frac{7}{(x-2)} + c}$$

Section 9

$$\left[\int \frac{7}{(x-2)^2}\,dx \text{ is determined using the algebraic substitution } u=(x-2)\text{, see Chapter 49}\right]$$

Problem 6. Find: $\displaystyle\int \frac{5x^2-2x-19}{(x+3)(x-1)^2}\,dx$

It was shown in Problem 6, page 57:

$$\frac{5x^2-2x-19}{(x+3)(x-1)^2} \equiv \frac{2}{(x+3)} + \frac{3}{(x-1)} - \frac{4}{(x-1)^2}$$

$$\text{Hence } \int \frac{5x^2-2x-19}{(x+3)(x-1)^2}\,dx$$

$$\equiv \int \left\{\frac{2}{(x+3)} + \frac{3}{(x-1)} - \frac{4}{(x-1)^2}\right\} dx$$

$$= \mathbf{2\ln(x+3) + 3\ln(x-1) + \frac{4}{(x-1)} + c}$$

$$\text{or } \quad \mathbf{\ln(x+3)^2\,(x-1)^3 + \frac{4}{(x-1)} + c}$$

Problem 7. Evaluate: $\displaystyle\int_{-2}^{1} \frac{3x^2+16x+15}{(x+3)^3}\,dx$, correct to 4 significant figures

It was shown in Problem 7, page 58:

$$\frac{3x^2+16x+15}{(x+3)^3} \equiv \frac{3}{(x+3)} - \frac{2}{(x+3)^2} - \frac{6}{(x+3)^3}$$

$$\text{Hence } \int \frac{3x^2+16x+15}{(x+3)^3}\,dx$$

$$\equiv \int_{-2}^{1} \left\{\frac{3}{(x+3)} - \frac{2}{(x+3)^2} - \frac{6}{(x+3)^3}\right\} dx$$

$$= \left[3\ln(x+3) + \frac{2}{(x+3)} + \frac{3}{(x+3)^2}\right]_{-2}^{1}$$

$$= \left(3\ln 4 + \frac{2}{4} + \frac{3}{16}\right) - \left(3\ln 1 + \frac{2}{1} + \frac{3}{1}\right)$$

$= \mathbf{-0.1536}$, correct to 4 significant figures.

Now try the following exercise

Exercise 182 Further problems on integration using partial fractions with repeated linear factors

In Problems 1 and 2, integrate with respect to x.

1. $\displaystyle\int \frac{4x-3}{(x+1)^2}\,dx$ $\quad\left[4\ln(x+1) + \frac{7}{(x+1)} + c\right]$

2. $\displaystyle\int \frac{5x^2-30x+44}{(x-2)^3}\,dx$
$\quad\left[5\ln(x-2) + \frac{10}{(x-2)} - \frac{2}{(x-2)^2} + c\right]$

In Problems 3 and 4, evaluate the definite integrals correct to 4 significant figures.

3. $\displaystyle\int_{1}^{2} \frac{x^2+7x+3}{x^2(x+3)}\,dx$ $\quad[1.663]$

4. $\displaystyle\int_{6}^{7} \frac{18+21x-x^2}{(x-5)(x+2)^2}\,dx$ $\quad[1.089]$

51.4 Worked problems on integration using partial fractions with quadratic factors

Problem 8. Find: $\displaystyle\int \frac{3+6x+4x^2-2x^3}{x^2(x^2+3)}\,dx$

It was shown in Problem 9, page 59:

$$\frac{3+6x+4x^2-2x^2}{x^2(x^2+3)} \equiv \frac{2}{x} + \frac{1}{x^2} + \frac{3-4x}{(x^2+3)}$$

$$\text{Thus } \int \frac{3+6x+4x^2-2x^3}{x^2(x^2+3)}\,dx$$

$$\equiv \int \left(\frac{2}{x} + \frac{1}{x^2} + \frac{3-4x}{(x^2+3)}\right) dx$$

$$= \int \left\{\frac{2}{x} + \frac{1}{x^2} + \frac{3}{(x^2+3)} - \frac{4x}{(x^2+3)}\right\} dx$$

$$\int \frac{3}{(x^2+3)}\,dx = 3\int \frac{1}{x^2+(\sqrt{3})^2}\,dx$$
$$= \frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}$$

from 12, Table 50.1, page 448.

$\int \frac{4x}{x^2+3}\,dx$ is determined using the algebraic substitutions $u=(x^2+3)$.

Hence $\int \left\{\frac{2}{x}+\frac{1}{x^2}+\frac{3}{(x^2+3)}-\frac{4x}{(x^2+3)}\right\}dx$

$$= 2\ln x - \frac{1}{x} + \frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 2\ln(x^2+3)+c$$
$$= \mathbf{\ln\left(\frac{x}{x^2+3}\right)^2 - \frac{1}{x} + \sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}} + c}$$

Problem 9. Determine: $\int \frac{1}{(x^2-a^2)}\,dx$

Let $\frac{1}{(x^2-a^2)} \equiv \frac{A}{(x-a)}+\frac{B}{(x+a)}$
$$\equiv \frac{A(x+a)+B(x-a)}{(x+a)(x-a)}$$

Equating the numerators gives:

$$1 \equiv A(x+a)+B(x-a)$$

Let $x=a$, then $A=\frac{1}{2a}$

and let $x=-a$,

then $B=-\frac{1}{2a}$

Hence $\int \frac{1}{(x^2-a^2)}\,dx \equiv \int \frac{1}{2a}\left[\frac{1}{(x-a)}-\frac{1}{(x+a)}\right]dx$

$$= \frac{1}{2a}[\ln(x-a)-\ln(x+a)]+c$$
$$= \mathbf{\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)+c}$$

Problem 10. Evaluate: $\int_3^4 \frac{3}{(x^2-4)}\,dx$, correct to 3 significant figures

From Problem 9,

$$\int_3^4 \frac{3}{(x^2-4)}\,dx = 3\left[\frac{1}{2(2)}\ln\left(\frac{x-2}{x+2}\right)\right]_3^4$$
$$= \frac{3}{4}\left[\ln\frac{2}{6}-\ln\frac{1}{5}\right]$$
$$= \frac{3}{4}\ln\frac{5}{3} = \mathbf{0.383}, \text{ correct to 3}$$

significant figures.

Problem 11. Determine: $\int \frac{1}{(a^2-x^2)}\,dx$

Using partial fractions, let

$$\frac{1}{(a^2-x^2)} \equiv \frac{1}{(a-x)(a+x)} \equiv \frac{A}{(a-x)}+\frac{B}{(a+x)}$$
$$\equiv \frac{A(a+x)+B(a-x)}{(a-x)(a+x)}$$

Then $1 \equiv A(a+x)+B(a-x)$

Let $x=a$ then $A=\frac{1}{2a}$. Let $x=-a$ then $B=\frac{1}{2a}$

Hence $\int \frac{1}{(a^2-x^2)}\,dx$

$$= \int \frac{1}{2a}\left[\frac{1}{(a-x)}+\frac{1}{(a+x)}\right]dx$$
$$= \frac{1}{2a}[-\ln(a-x)+\ln(a+x)]+c$$
$$= \mathbf{\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)+c}$$

Problem 12. Evaluate: $\int_0^2 \frac{5}{(9-x^2)}\,dx$, correct to 4 decimal places

From Problem 11,

$$\int_0^2 \frac{5}{(9-x^2)}\,dx = 5\left[\frac{1}{2(3)}\ln\left(\frac{3+x}{3-x}\right)\right]_0^2$$
$$= \frac{5}{6}\left[\ln\frac{5}{1}-\ln 1\right] = \mathbf{1.3412},$$

correct to 4 decimal places

Now try the following exercise

Exercise 183 Further problems on integration using partial fractions with quadratic factors

1. Determine $\int \frac{x^2 - x - 13}{(x^2+7)(x-2)}\,dx$

$$\left[\begin{array}{c}\ln(x^2+7) + \frac{3}{\sqrt{7}}\tan^{-1}\frac{x}{\sqrt{7}} \\ -\ln(x-2)+c\end{array}\right]$$

In Problems 2 to 4, evaluate the definite integrals correct to 4 significant figures.

2. $\int_5^6 \frac{6x-5}{(x-4)(x^2+3)}\,dx$ [0.5880]

3. $\int_1^2 \frac{4}{(16-x^2)}\,dx$ [0.2939]

4. $\int_4^5 \frac{2}{(x^2-9)}\,dx$ [0.1865]

Chapter 52

The $t = \tan\frac{\theta}{2}$ substitution

52.1 Introduction

Integrals of the form $\int \frac{1}{a\cos\theta + b\sin\theta + c}\,d\theta$, where a, b and c are constants, may be determined by using the substitution $t = \tan\frac{\theta}{2}$. The reason is explained below.

If angle A in the right-angled triangle ABC shown in Fig. 52.1 is made equal to $\frac{\theta}{2}$ then, since tangent $= \frac{\text{opposite}}{\text{adjacent}}$, if $BC = t$ and $AB = 1$, then $\tan\frac{\theta}{2} = t$.

By Pythagoras' theorem, $AC = \sqrt{1+t^2}$

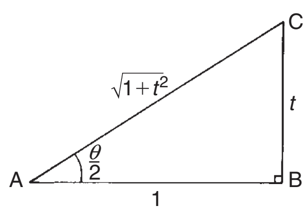

Figure 52.1

Therefore $\sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}}$ and $\cos\frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}$

Since $\sin 2x = 2\sin x\cos x$ (from double angle formulae, Chapter 27), then

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)$$

i.e. $$\mathbf{\sin\theta = \frac{2t}{(1+t^2)}} \qquad (1)$$

Since $$\cos 2x = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$= \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{1}{\sqrt{1+t^2}}\right)^2$$

i.e. $$\mathbf{\cos\theta = \frac{1-t^2}{1+t^2}} \qquad (2)$$

Also, since $t = \tan\frac{\theta}{2}$

$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2} = \frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)$ from trigonometric identities,

i.e. $$\frac{dt}{d\theta} = \frac{1}{2}(1+t^2)$$

from which, $$\mathbf{d\theta = \frac{2\,dt}{1+t^2}} \qquad (3)$$

Equations (1), (2) and (3) are used to determine integrals of the form $\int \frac{1}{a\cos\theta + b\sin\theta + c}\,d\theta$ where a, b or c may be zero.

52.2 Worked problems on the $t = \tan\frac{\theta}{2}$ substitution

Problem 1. Determine: $\int \frac{d\theta}{\sin\theta}$

If $t = \tan\frac{\theta}{2}$ then $\sin\theta = \frac{2t}{1+t^2}$ and $d\theta = \frac{2\,dt}{1+t^2}$ from equations (1) and (3).

Thus $$\int \frac{d\theta}{\sin\theta} = \int \frac{1}{\sin\theta}\,d\theta$$

$$= \int \frac{1}{\frac{2t}{1+t^2}} \left(\frac{2\,dt}{1+t^2}\right)$$

$$= \int \frac{1}{t} dt = \ln t + c$$

Hence $$\int \frac{d\theta}{\sin\theta} = \mathbf{\ln\left(\tan\frac{\theta}{2}\right) + c}$$

Problem 2. Determine: $\int \frac{dx}{\cos x}$

If $t = \tan\frac{x}{2}$ then $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2\,dt}{1+t^2}$ from equations (2) and (3).

Thus $$\int \frac{dx}{\cos x} = \int \frac{1}{\frac{1-t^2}{1+t^2}} \left(\frac{2\,dt}{1+t^2}\right)$$

$$= \int \frac{2}{1-t^2} dt$$

$\frac{2}{1-t^2}$ may be resolved into partial fractions (see Chapter 7).

Let $$\frac{2}{1-t^2} = \frac{2}{(1-t)(1+t)}$$

$$= \frac{A}{(1-t)} + \frac{B}{(1+t)}$$

$$= \frac{A(1+t)+B(1-t)}{(1-t)(1+t)}$$

Hence $$2 = A(1+t) + B(1-t)$$

When $t = 1$, $2 = 2A$, from which, $A = 1$

When $t = -1$, $2 = 2B$, from which, $B = 1$

Hence $$\int \frac{2dt}{1-t^2} = \int \frac{1}{(1-t)} + \frac{1}{(1+t)} dt$$

$$= -\ln(1-t) + \ln(1+t) + c$$

$$= \ln\left\{\frac{(1+t)}{(1-t)}\right\} + c$$

Thus $$\int \frac{dx}{\cos x} = \mathbf{\ln\left\{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right\} + c}$$

Note that since $\tan\frac{\pi}{4} = 1$, the above result may be written as:

$$\int \frac{dx}{\cos x} = \ln\left\{\frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}}\right\} + c$$

$$= \mathbf{\ln\left\{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\} + c}$$

from compound angles, Chapter 27

Problem 3. Determine: $\int \frac{dx}{1+\cos x}$

If $t = \tan\frac{x}{2}$ then $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$ from equations (2) and (3).

Thus $$\int \frac{dx}{1+\cos x} = \int \frac{1}{1+\cos x} dx$$

$$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \left(\frac{2\,dt}{1+t^2}\right)$$

$$= \int \frac{1}{\frac{(1+t^2)+(1-t^2)}{1+t^2}} \left(\frac{2\,dt}{1+t^2}\right)$$

$$= \int dt$$

Hence $$\int \frac{dx}{1+\cos x} = t + c = \mathbf{\tan\frac{x}{2} + c}$$

Problem 4. Determine: $\int \frac{d\theta}{5+4\cos\theta}$

If $t = \tan\frac{\theta}{2}$ then $\cos\theta = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2\,dt}{1+t^2}$ from equations (2) and (3).

Thus $$\int \frac{d\theta}{5+4\cos\theta} = \int \frac{\left(\frac{2\,dt}{1+t^2}\right)}{5+4\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int \frac{\left(\frac{2\,dt}{1+t^2}\right)}{\frac{5(1+t^2)+4(1-t^2)}{1+t^2}}$$

$$= 2\int \frac{dt}{t^2+9} = 2\int \frac{dt}{t^2+3^2}$$

$$= 2\left(\frac{1}{3}\tan^{-1}\frac{t}{3}\right) + c,$$

from 12 of Table 50.1, page 448. Hence

$$\int \frac{d\theta}{5+4\cos\theta} = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{\theta}{2}\right) + c$$

Now try the following exercise

Exercise 184 Further problems on the $t = \tan\frac{\theta}{2}$ substitution

Integrate the following with respect to the variable:

1. $\displaystyle\int \frac{d\theta}{1+\sin\theta}$ $\left[\dfrac{-2}{1+\tan\frac{\theta}{2}} + c\right]$

2. $\displaystyle\int \frac{dx}{1-\cos x+\sin x}$ $\left[\ln\left\{\dfrac{\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right\} + c\right]$

3. $\displaystyle\int \frac{d\alpha}{3+2\cos\alpha}$ $\left[\dfrac{2}{\sqrt{5}}\tan^{-1}\left(\dfrac{1}{\sqrt{5}}\tan\dfrac{\alpha}{2}\right) + c\right]$

4. $\displaystyle\int \frac{dx}{3\sin x-4\cos x}$ $\left[\dfrac{1}{5}\ln\left\{\dfrac{2\tan\frac{x}{2}-1}{\tan\frac{x}{2}+2}\right\} + c\right]$

52.3 Further worked problems on the $t = \tan\frac{\theta}{2}$ substitution

Problem 5. Determine: $\displaystyle\int \frac{dx}{\sin x+\cos x}$

If $t = \tan\frac{x}{2}$ then $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$ and $dx = \dfrac{2\,dt}{1+t^2}$ from equations (1), (2) and (3).

Thus

$$\int \frac{dx}{\sin x+\cos x} = \int \frac{\dfrac{2\,dt}{1+t^2}}{\left(\dfrac{2t}{1+t^2}\right)+\left(\dfrac{1-t^2}{1+t^2}\right)}$$

$$= \int \frac{\dfrac{2\,dt}{1+t^2}}{\dfrac{2t+1-t^2}{1+t^2}} = \int \frac{2\,dt}{1+2t-t^2}$$

$$= \int \frac{-2\,dt}{t^2-2t-1} = \int \frac{-2\,dt}{(t-1)^2-2}$$

$$= \int \frac{2\,dt}{(\sqrt{2})^2-(t-1)^2}$$

$$= 2\left[\frac{1}{2\sqrt{2}}\ln\left\{\frac{\sqrt{2}+(t-1)}{\sqrt{2}-(t-1)}\right\}\right] + c$$

(see problem 11, Chapter 51, page 458),

i.e. $\displaystyle\int \frac{dx}{\sin x+\cos x}$

$$= \frac{1}{\sqrt{2}}\ln\left\{\frac{\sqrt{2}-1+\tan\frac{x}{2}}{\sqrt{2}+1-\tan\frac{x}{2}}\right\} + c$$

Problem 6. Determine: $\displaystyle\int \frac{dx}{7-3\sin x+6\cos x}$

From equations (1) and (3),

$$\int \frac{dx}{7-3\sin x+6\cos x}$$

$$= \int \frac{\dfrac{2\,dt}{1+t^2}}{7-3\left(\dfrac{2t}{1+t^2}\right)+6\left(\dfrac{1-t^2}{1+t^2}\right)}$$

$$= \int \frac{\dfrac{2\,dt}{1+t^2}}{\dfrac{7(1+t^2)-3(2t)+6(1-t^2)}{1+t^2}}$$

$$= \int \frac{2\,dt}{7+7t^2-6t+6-6t^2}$$

$$= \int \frac{2\,dt}{t^2-6t+13} = \int \frac{2\,dt}{(t-3)^2+2^2}$$

$$= 2\left[\frac{1}{2}\tan^{-1}\left(\frac{t-3}{2}\right)\right] + c$$

from 12, Table 50.1, page 448. Hence

$$\int \frac{dx}{7 - 3\sin x + 6\cos x} = \tan^{-1}\left(\frac{\tan\frac{x}{2} - 3}{2}\right) + c$$

Problem 7. Determine: $\displaystyle\int \frac{d\theta}{4\cos\theta + 3\sin\theta}$

From equations (1) to (3),

$$\int \frac{d\theta}{4\cos\theta + 3\sin\theta} = \int \frac{\frac{2\,dt}{1+t^2}}{4\left(\frac{1-t^2}{1+t^2}\right) + 3\left(\frac{2t}{1+t^2}\right)}$$

$$= \int \frac{2\,dt}{4 - 4t^2 + 6t} = \int \frac{dt}{2 + 3t - 2t^2}$$

$$= -\frac{1}{2}\int \frac{dt}{t^2 - \frac{3}{2}t - 1}$$

$$= -\frac{1}{2}\int \frac{dt}{\left(t - \frac{3}{4}\right)^2 - \frac{25}{16}}$$

$$= \frac{1}{2}\int \frac{dt}{\left(\frac{5}{4}\right)^2 - \left(t - \frac{3}{4}\right)^2}$$

$$= \frac{1}{2}\left[\frac{1}{2\left(\frac{5}{4}\right)} \ln\left\{\frac{\frac{5}{4} + \left(t - \frac{3}{4}\right)}{\frac{5}{4} - \left(t - \frac{3}{4}\right)}\right\}\right] + c$$

from problem 11, Chapter 51, page 458,

$$= \frac{1}{5}\ln\left\{\frac{\frac{1}{2} + t}{2 - t}\right\} + c$$

Hence $\displaystyle\int \frac{d\theta}{4\cos\theta + 3\sin\theta}$

$$= \mathbf{\frac{1}{5}\ln\left\{\frac{\frac{1}{2} + \tan\frac{\theta}{2}}{2 - \tan\frac{\theta}{2}}\right\} + c}$$

or $\displaystyle\mathbf{\frac{1}{5}\ln\left\{\frac{1 + 2\tan\frac{\theta}{2}}{4 - 2\tan\frac{\theta}{2}}\right\} + c}$

Now try the following exercise

Exercise 185 Further problems on the $t = \tan\frac{\theta}{2}$ substitution

In Problems 1 to 4, integrate with respect to the variable.

1. $\displaystyle\int \frac{d\theta}{5 + 4\sin\theta}$ $\left[\frac{2}{3}\tan^{-1}\left(\frac{5\tan\frac{\theta}{2} + 4}{3}\right) + c\right]$

2. $\displaystyle\int \frac{dx}{1 + 2\sin x}$ $\left[\frac{1}{\sqrt{3}}\ln\left\{\frac{\tan\frac{x}{2} + 2 - \sqrt{3}}{\tan\frac{x}{2} + 2 + \sqrt{3}}\right\} + c\right]$

3. $\displaystyle\int \frac{dp}{3 - 4\sin p + 2\cos p}$ $\left[\frac{1}{\sqrt{11}}\ln\left\{\frac{\tan\frac{p}{2} - 4 - \sqrt{11}}{\tan\frac{p}{2} - 4 + \sqrt{11}}\right\} + c\right]$

4. $\displaystyle\int \frac{d\theta}{3 - 4\sin\theta}$ $\left[\frac{1}{\sqrt{7}}\ln\left\{\frac{3\tan\frac{\theta}{2} - 4 - \sqrt{7}}{3\tan\frac{\theta}{2} - 4 + \sqrt{7}}\right\} + c\right]$

5. Show that
$$\int \frac{dt}{1 + 3\cos t} = \frac{1}{2\sqrt{2}}\ln\left\{\frac{\sqrt{2} + \tan\frac{t}{2}}{\sqrt{2} - \tan\frac{t}{2}}\right\} + c$$

6. Show that $\displaystyle\int_0^{\pi/3} \frac{3\,d\theta}{\cos\theta} = 3.95$, correct to 3 significant figures.

7. Show that $\displaystyle\int_0^{\pi/2} \frac{d\theta}{2 + \cos\theta} = \frac{\pi}{3\sqrt{3}}$

Chapter 53

Integration by parts

53.1 Introduction

From the product rule of differentiation:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

where u and v are both functions of x.

Rearranging gives: $u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$

Integrating both sides with respect to x gives:

$$\int u\frac{dv}{dx}dx = \int \frac{d}{dx}(uv)dx - \int v\frac{du}{dx}dx$$

ie $$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

or $$\int u\,dv = uv - \int v\,du$$

This is known as the **integration by parts formula** and provides a method of integrating such products of simple functions as $\int xe^x dx$, $\int t \sin t\, dt$, $\int e^\theta \cos\theta\, d\theta$ and $\int x \ln x\, dx$.

Given a product of two terms to integrate the initial choice is: 'which part to make equal to u' and 'which part to make equal to dv'. The choice must be such that the 'u part' becomes a constant after successive differentiation and the 'dv part' can be integrated from standard integrals. Invariable, the following rule holds: 'If a product to be integrated contains an algebraic term (such as x, t^2 or 3θ) then this term is chosen as the u part. The one exception to this rule is when a '$\ln x$' term is involved; in this case $\ln x$ is chosen as the 'u part'.

53.2 Worked problems on integration by parts

Problem 1. Determine: $\int x \cos x\, dx$

From the integration by parts formula,

$$\int u\,dv = uv - \int v\,du$$

Let $u = x$, from which $\frac{du}{dx} = 1$, i.e. $du = dx$ and let $dv = \cos x\, dx$, from which $v = \int \cos x\, dx = \sin x$.

Expressions for u, du and v are now substituted into the 'by parts' formula as shown below.

$$\int u \; dv = u \; v - \int v \; du$$
$$\int (x)\; (\cos x\, dx) = (x)(\sin x) - \int (\sin x)(dx)$$

i.e. $$\int x \cos x\, dx = x\sin x - (-\cos x) + c$$

$$= \mathbf{x \sin x + \cos x + c}$$

[This result may be checked by differentiating the right hand side,

i.e. $\frac{d}{dx}(x \sin x + \cos x + c)$

$= [(x)(\cos x) + (\sin x)(1)] - \sin x + 0$

using the product rule

$= x\cos x$, which is the function being integrated

Problem 2. Find: $\int 3te^{2t}\,dt$

Let $u = 3t$, from which, $\frac{du}{dt} = 3$, i.e. $du = 3\,dt$ and let $dv = e^{2t}\,dt$, from which, $v = \int e^{2t} dt = \frac{1}{2}e^{2t}$

Substituting into $\int u\,dv = u\,v - \int v\,du$ gives:

$$\int 3te^{2t}dt = (3t)\left(\frac{1}{2}e^{2t}\right) - \int \left(\frac{1}{2}e^{2t}\right)(3\,dt)$$

$$= \frac{3}{2}te^{2t} - \frac{3}{2}\int e^{2t}dt$$

$$= \frac{3}{2}te^{2t} - \frac{3}{2}\left(\frac{e^{2t}}{2}\right) + c$$

Hence $\boldsymbol{\int 3te^{2t}\,dt = \frac{3}{2}e^{2t}\left(t - \frac{1}{2}\right) + c,}$

which may be checked by differentiating.

Problem 3. Evaluate $\int_0^{\frac{\pi}{2}} 2\theta \sin\theta\,d\theta$

Let $u = 2\theta$, from which, $\frac{du}{d\theta} = 2$, i.e. $du = 2\,d\theta$ and let $dv = \sin\theta\,d\theta$, from which,

$$v = \int \sin\theta\,d\theta = -\cos\theta$$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int 2\theta\sin\theta\,d\theta = (2\theta)(-\cos\theta) - \int(-\cos\theta)(2\,d\theta)$$

$$= -2\theta\cos\theta + 2\int \cos\theta\,d\theta$$

$$= -2\theta\cos\theta + 2\sin\theta + c$$

Hence $\int_0^{\frac{\pi}{2}} 2\theta\sin\theta\,d\theta$

$$= \Big[2\theta\cos\theta + 2\sin\theta\Big]_0^{\frac{\pi}{2}}$$

$$= \left[-2\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2} + 2\sin\frac{\pi}{2}\right] - [0 + 2\sin 0]$$

$$= (-0 + 2) - (0 + 0) = \mathbf{2}$$

since $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$

Problem 4. Evaluate: $\int_0^1 5xe^{4x}dx$, correct to 3 significant figures

Let $u = 5x$, from which $\frac{du}{dx} = 5$, i.e. $du = 5\,dx$ and let $dv = e^{4x}\,dx$, from which, $v = \int e^{4x}dx = \frac{1}{4}e^{4x}$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int 5xe^{4x}\,dx = (5x)\left(\frac{e^{4x}}{4}\right) - \int\left(\frac{e^{4x}}{4}\right)(5\,dx)$$

$$= \frac{5}{4}xe^{4x} - \frac{5}{4}\int e^{4x}\,dx$$

$$= \frac{5}{4}xe^{4x} - \frac{5}{4}\left(\frac{e^{4x}}{4}\right) + c$$

$$= \frac{5}{4}e^{4x}\left(x - \frac{1}{4}\right) + c$$

Hence $\int_0^1 5xe^{4x}\,dx$

$$= \left[\frac{5}{4}e^{4x}\left(x - \frac{1}{4}\right)\right]_0^1$$

$$= \left[\frac{5}{4}e^4\left(1 - \frac{1}{4}\right)\right] - \left[\frac{5}{4}e^0\left(0 - \frac{1}{4}\right)\right]$$

$$= \left(\frac{15}{16}e^4\right) - \left(-\frac{5}{16}\right)$$

$$= 51.186 + 0.313 = 51.499 = \mathbf{51.5},$$

correct to 3 significant figures.

Problem 5. Determine: $\int x^2\sin x\,dx$

Let $u = x^2$, from which, $\frac{du}{dx} = 2x$, i.e. $du = 2x\,dx$, and let $dv = \sin x\,dx$, from which, $v = \int \sin x\,dx = -\cos x$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int x^2\sin x\,dx = (x^2)(-\cos x) - \int(-\cos x)(2x\,dx)$$

$$= -x^2\cos x + 2\left[\int x\cos x\,dx\right]$$

The integral, $\int x\cos x\,dx$, is not a 'standard integral' and it can only be determined by using the integration by parts formula again.

From Problem 1, $\int x\cos x\,dx = x\sin x + \cos x$

Hence $\int x^2 \sin x\,dx$

$$= -x^2\cos x + 2\{x\sin x + \cos x\} + c$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

$$= \mathbf{(2 - x^2)\cos x + 2x\sin x + c}$$

In general, if the algebraic term of a product is of power n, then the integration by parts formula is applied n times.

Now try the following exercise

Exercise 186 Further problems on integration by parts

Determine the integrals in Problems 1 to 5 using integration by parts.

1. $\int xe^{2x}dx$ $\quad\left[\frac{e^{2x}}{2}\left(x-\frac{1}{2}\right)+c\right]$

2. $\int \frac{4x}{e^{3x}}dx$ $\quad\left[-\frac{4}{3}e^{-3x}\left(x+\frac{1}{3}\right)+c\right]$

3. $\int x\sin x\,dx$ $\quad[-x\cos x + \sin x + c]$

4. $\int 5\theta\cos 2\theta\,d\theta$ $\quad\left[\frac{5}{2}\left(\theta\sin 2\theta + \frac{1}{2}\cos 2\theta\right)+c\right]$

5. $\int 3t^2e^{2t}dt$ $\quad\left[\frac{3}{2}e^{2t}\left(t^2 - t + \frac{1}{2}\right)+c\right]$

Evaluate the integrals in Problems 6 to 9, correct to 4 significant figures.

6. $\int_0^2 2xe^x\,dx$ $\quad[16.78]$

7. $\int_0^{\frac{\pi}{4}} x\sin 2x\,dx$ $\quad[0.2500]$

8. $\int_0^{\frac{\pi}{2}} t^2\cos t\,dt$ $\quad[0.4674]$

9. $\int_1^2 3x^2e^{\frac{x}{2}}\,dx$ $\quad[15.78]$

53.3 Further worked problems on integration by parts

Problem 6. Find: $\int x\ln x\,dx$

The logarithmic function is chosen as the 'u part' Thus when $u = \ln x$, then $\frac{du}{dx} = \frac{1}{x}$ i.e. $du = \frac{dx}{x}$

Letting $dv = x\,dx$ gives $v = \int x\,dx = \frac{x^2}{2}$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int x\ln x\,dx = (\ln x)\left(\frac{x^2}{2}\right) - \int\left(\frac{x^2}{2}\right)\frac{dx}{x}$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

Hence $\int x\ln x\,dx = \mathbf{\frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c}$

or $\mathbf{\frac{x^2}{4}(2\ln x - 1) + c}$

Problem 7. Determine: $\int \ln x\,dx$

$\int \ln x\,dx$ is the same as $\int (1)\ln x\,dx$

Let $u = \ln x$, from which, $\frac{du}{dx} = \frac{1}{x}$ i.e. $du = \frac{dx}{x}$ and let $dv = 1\,dx$, from which, $v = \int 1\,dx = x$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int \ln x\,dx = (\ln x)(x) - \int x\frac{dx}{x}$$

$$= x\ln x - \int dx = x\ln x - x + c$$

Hence $\int \mathbf{\ln x\,dx = x(\ln x - 1) + c}$

Problem 8. Evaluate: $\int_1^9 \sqrt{x}\ln x\,dx$, correct to 3 significant figures

Let $u = \ln x$, from which $du = \dfrac{dx}{x}$

and let $dv = \sqrt{x}\,dx = x^{\frac{1}{2}}\,dx$, from which,

$$v = \int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}}$$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int \sqrt{x}\ln x\,dx = (\ln x)\left(\frac{2}{3}x^{\frac{3}{2}}\right) - \int \left(\frac{2}{3}x^{\frac{3}{2}}\right)\left(\frac{dx}{x}\right)$$

$$= \frac{2}{3}\sqrt{x^3}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}\,dx$$

$$= \frac{2}{3}\sqrt{x^3}\ln x - \frac{2}{3}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + c$$

$$= \frac{2}{3}\sqrt{x^3}\left[\ln x - \frac{2}{3}\right] + c$$

Hence $\displaystyle\int_1^9 \sqrt{x}\ln x\,dx = \left[\frac{2}{3}\sqrt{x^3}\left(\ln x - \frac{2}{3}\right)\right]_1^9$

$$= \left[\frac{2}{3}\sqrt{9^3}\left(\ln 9 - \frac{2}{3}\right)\right] - \left[\frac{2}{3}\sqrt{1^3}\left(\ln 1 - \frac{2}{3}\right)\right]$$

$$= \left[18\left(\ln 9 - \frac{2}{3}\right)\right] - \left[\frac{2}{3}\left(0 - \frac{2}{3}\right)\right]$$

$= 27.550 + 0.444 = 27.994 = \mathbf{28.0}$,

correct to 3 significant figures.

Problem 9. Find: $\int e^{ax}\cos bx\,dx$

When integrating a product of an exponential and a sine or cosine function it is immaterial which part is made equal to 'u'.

Let $u = e^{ax}$, from which $\dfrac{du}{dx} = ae^{ax}$, i.e. $du = ae^{ax}\,dx$

and let $dv = \cos bx\,dx$, from which,

$$v = \int \cos bx\,dx = \frac{1}{b}\sin bx$$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int e^{ax}\cos bx\,dx$$

$$= (e^{ax})\left(\frac{1}{b}\sin bx\right) - \int \left(\frac{1}{b}\sin bx\right)(ae^{ax}\,dx)$$

$$= \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[\int e^{ax}\sin bx\,dx\right] \qquad (1)$$

$\int e^{ax}\sin bx\,dx$ is now determined separately using integration by parts again:

Let $u = e^{ax}$ then $du = ae^{ax}\,dx$, and let $dv = \sin bx\,dx$, from which

$$v = \int \sin bx\,dx = -\frac{1}{b}\cos bx$$

Substituting into the integration by parts formula gives:

$$\int e^{ax}\sin bx\,dx = (e^{ax})\left(-\frac{1}{b}\cos bx\right) - \int \left(-\frac{1}{b}\cos bx\right)(ae^{ax}\,dx)$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx$$

Substituting this result into equation (1) gives:

$$\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx\right]$$

$$= \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx$$

The integral on the far right of this equation is the same as the integral on the left hand side and thus they may be combined.

$$\int e^{ax}\cos bx\,dx + \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx$$

i.e. $\displaystyle\left(1 + \frac{a^2}{b^2}\right)\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx$

i.e. $\displaystyle\left(\frac{b^2 + a^2}{b^2}\right)\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{b^2}(b\sin bx + a\cos bx)$

Hence $\int e^{ax} \cos bx\, dx$

$$= \left(\frac{b^2}{b^2 + a^2}\right)\left(\frac{e^{ax}}{b^2}\right)(b \sin bx + a \cos bx)$$

$$= \frac{e^{ax}}{a^2 + b^2}(b \sin bx + a \cos bx) + c$$

Using a similar method to above, that is, integrating by parts twice, the following result may be proved:

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c \qquad (2)$$

Problem 10. Evaluate $\int_0^{\frac{\pi}{4}} e^t \sin 2t\, dt$, correct to 4 decimal places

Comparing $\int e^t \sin 2t\, dt$ with $\int e^{ax} \sin bx\, dx$ shows that $x = t$, $a = 1$ and $b = 2$.
Hence, substituting into equation (2) gives:

$$\int_0^{\frac{\pi}{4}} e^t \sin 2t\, dt$$

$$= \left[\frac{e^t}{1^2 + 2^2}(1 \sin 2t - 2 \cos 2t)\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{e^{\frac{\pi}{4}}}{5}\left(\sin 2\left(\frac{\pi}{4}\right) - 2 \cos 2\left(\frac{\pi}{4}\right)\right)\right] - \left[\frac{e^0}{5}(\sin 0 - 2 \cos 0)\right]$$

$$= \left[\frac{e^{\frac{\pi}{4}}}{5}(1 - 0)\right] - \left[\frac{1}{5}(0 - 2)\right] = \frac{e^{\frac{\pi}{4}}}{5} + \frac{2}{5}$$

$= \mathbf{0.8387}$, correct to 4 decimal places

Now try the following exercise

Exercise 187 Further problems on integration by parts

Determine the integrals in Problems 1 to 5 using integration by parts.

1. $\int 2x^2 \ln x\, dx$ $\quad\left[\frac{2}{3}x^3\left(\ln x - \frac{1}{3}\right) + c\right]$

2. $\int 2 \ln 3x\, dx$ $\quad[2x(\ln 3x - 1) + c]$

3. $\int x^2 \sin 3x\, dx$ $\quad\left[\frac{\cos 3x}{27}(2 - 9x^2) + \frac{2}{9}x \sin 3x + c\right]$

4. $\int 2e^{5x} \cos 2x\, dx$ $\quad\left[\frac{2}{29}e^{5x}(2 \sin 2x + 5 \cos 2x) + c\right]$

5. $\int 2\theta \sec^2 \theta\, d\theta$ $\quad[2[\theta \tan \theta - \ln(\sec \theta)] + c]$

Evaluate the integrals in Problems 6 to 9, correct to 4 significant figures.

6. $\int_1^2 x \ln x\, dx$ $\quad[0.6363]$

7. $\int_0^1 2e^{3x} \sin 2x\, dx$ $\quad[11.31]$

8. $\int_0^{\frac{\pi}{2}} e^t \cos 3t\, dt$ $\quad[-1.543]$

9. $\int_1^4 \sqrt{x^3} \ln x\, dx$ $\quad[12.78]$

10. In determining a Fourier series to represent $f(x) = x$ in the range $-\pi$ to π, Fourier coefficients are given by:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx\, dx$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\, dx$$

where n is a positive integer. Show by using integration by parts that $a_n = 0$ and $b_n = -\frac{2}{n} \cos n\pi$

11. The equations:

$$C = \int_0^1 e^{-0.4\theta} \cos 1.2\theta\, d\theta$$

and $$S = \int_0^1 e^{-0.4\theta} \sin 1.2\theta\, d\theta$$

are involved in the study of damped oscillations. Determine the values of C and S.
$[C = 0.66, S = 0.41]$

Section 9

Chapter 54

Numerical integration

54.1 Introduction

Even with advanced methods of integration there are many mathematical functions which cannot be integrated by analytical methods and thus approximate methods have then to be used. Approximate methods of definite integrals may be determined by what is termed **numerical integration**.

It may be shown that determining the value of a definite integral is, in fact, finding the area between a curve, the horizontal axis and the specified ordinates. Three methods of finding approximate areas under curves are the trapezoidal rule, the mid-ordinate rule and Simpson's rule, and these rules are used as a basis for numerical integration.

54.2 The trapezoidal rule

Let a required definite integral be denoted by $\int_a^b y\,dx$ and be represented by the area under the graph of $y = f(x)$ between the limits $x = a$ and $x = b$ as shown in Fig. 54.1.

Let the range of integration be divided into n equal intervals each of width d, such that $nd = b - a$, i.e. $d = \dfrac{b-a}{n}$

The ordinates are labelled $y_1, y_2, y_3, \ldots, y_{n+1}$ as shown.

An approximation to the area under the curve may be determined by joining the tops of the ordinates by straight lines. Each interval is thus a trapezium, and since the area of a trapezium is given by:

$$\text{area} = \frac{1}{2}(\text{sum of parallel sides}) (\text{perpendicular distance between them})$$

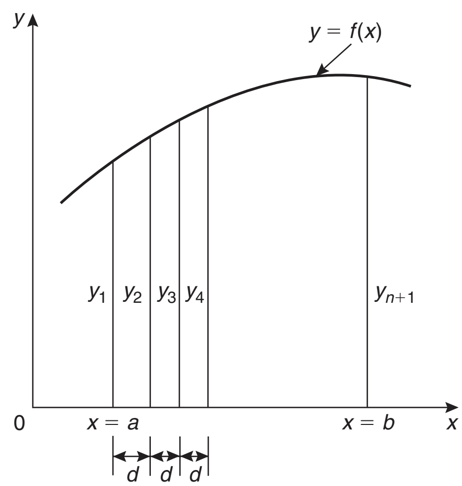

Figure 54.1

then

$$\int_a^b y\,dx \approx \frac{1}{2}(y_1 + y_2)d + \frac{1}{2}(y_2 + y_3)d + \frac{1}{2}(y_3 + y_4)d + \cdots + \frac{1}{2}(y_n + y_{n+1})d$$

$$\approx d\left[\frac{1}{2}y_1 + y_2 + y_3 + y_4 + \cdots + y_n + \frac{1}{2}y_{n+1}\right]$$

i.e. **the trapezoidal rule states**:

$$\int_a^b y\,dx \approx \left(\begin{array}{c}\text{width of}\\ \text{interval}\end{array}\right)\left\{\frac{1}{2}\left(\begin{array}{c}\text{first + last}\\ \text{ordinate}\end{array}\right) + \left(\begin{array}{c}\text{sum of}\\ \text{remaining}\\ \text{ordinates}\end{array}\right)\right\} \quad (1)$$

Problem 1. (a) Use integration to evaluate, correct to 3 decimal places, $\displaystyle\int_1^3 \frac{2}{\sqrt{x}}\,dx$
(b) Use the trapezoidal rule with 4 intervals to evaluate the integral in part (a), correct to 3 decimal places

(a) $\int_1^3 \frac{2}{\sqrt{x}}\,dx = \int_1^3 2x^{-\frac{1}{2}}\,dx$

$$= \left[\frac{2x^{\left(-\frac{1}{2}\right)+1}}{-\frac{1}{2}+1}\right]_1^3 = \left[4x^{\frac{1}{2}}\right]_1^3$$

$$= 4\left[\sqrt{x}\right]_1^3 = 4\left[\sqrt{3}-\sqrt{1}\right]$$

$= \mathbf{2.928}$, correct to 3 decimal places.

(b) The range of integration is the difference between the upper and lower limits, i.e. $3-1=2$. Using the trapezoidal rule with 4 intervals gives an interval width $d = \frac{3-1}{4} = 0.5$ and ordinates situated at 1.0, 1.5, 2.0, 2.5 and 3.0. Corresponding values of $\frac{2}{\sqrt{x}}$ are shown in the table below, each correct to 4 decimal places (which is one more decimal place than required in the problem).

x	$\frac{2}{\sqrt{x}}$
1.0	2.0000
1.5	1.6330
2.0	1.4142
2.5	1.2649
3.0	1.1547

From equation (1):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx (0.5)\left\{\frac{1}{2}(2.0000+1.1547) + 1.6330 + 1.4142 + 1.2649\right\}$$

$= \mathbf{2.945}$, correct to 3 decimal places.

This problem demonstrates that even with just 4 intervals a close approximation to the true value of 2.928 (correct to 3 decimal places) is obtained using the trapezoidal rule.

Problem 2. Use the trapezoidal rule with 8 intervals to evaluate $\int_1^3 \frac{2}{\sqrt{x}}\,dx$, correct to 3 decimal places

With 8 intervals, the width of each is $\frac{3-1}{8}$ i.e. 0.25 giving ordinates at 1.00, 1.25, 1.50, 1.75, 2.00, 2.25, 2.50, 2.75 and 3.00. Corresponding values of $\frac{2}{\sqrt{x}}$ are shown in the table below:

x	$\frac{2}{\sqrt{x}}$
1.00	2.000
1.25	1.7889
1.50	1.6330
1.75	1.5119
2.00	1.4142
2.25	1.3333
2.50	1.2649
2.75	1.2060
3.00	1.1547

From equation (1):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx (0.25)\left\{\frac{1}{2}(2.000+1.1547) + 1.7889 + 1.6330 + 1.5119 + 1.4142 + 1.3333 + 1.2649 + 1.2060\right\}$$

$= \mathbf{2.932}$, correct to 3 decimal places

This problem demonstrates that the greater the number of intervals chosen (i.e. the smaller the interval width) the more accurate will be the value of the definite integral. The exact value is found when the number of intervals is infinite, which is what the process of integration is based upon.

Problem 3. Use the trapezoidal rule to evaluate $\int_0^{\pi/2} \frac{1}{1+\sin x}\,dx$ using 6 intervals. Give the answer correct to 4 significant figures

With 6 intervals, each will have a width of $\frac{\frac{\pi}{2}-0}{6}$ i.e. $\frac{\pi}{12}$ rad (or 15°) and the ordinates occur at 0,

$\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}$ and $\frac{\pi}{2}$. Corresponding values of $\frac{1}{1+\sin x}$ are shown in the table below:

x	$\frac{1}{1+\sin x}$
0	1.0000
$\frac{\pi}{12}$ (or 15°)	0.79440
$\frac{\pi}{6}$ (or 30°)	0.66667
$\frac{\pi}{4}$ (or 45°)	0.58579
$\frac{\pi}{3}$ (or 60°)	0.53590
$\frac{5\pi}{12}$ (or 75°)	0.50867
$\frac{\pi}{2}$ (or 90°)	0.50000

From equation (1):

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx \approx \left(\frac{\pi}{12}\right)\left\{\frac{1}{2}(1.00000+0.50000) + 0.79440 + 0.66667 + 0.58579 + 0.53590 + 0.50867\right\}$$

$$= \mathbf{1.006}, \text{ correct to 4 significant figures}$$

Now try the following exercise

Exercise 188 Further problems on the trapezoidal rule

Evaluate the following definite integrals using the **trapezoidal rule**, giving the answers correct to 3 decimal places:

1. $\int_0^1 \frac{2}{1+x^2}\,dx$ (Use 8 intervals) [1.569]

2. $\int_1^3 2\ln 3x\,dx$ (Use 8 intervals) [6.979]

3. $\int_0^{\pi/3} \sqrt{\sin\theta}\,d\theta$ (Use 6 intervals) [0.672]

4. $\int_0^{1.4} e^{-x^2}\,dx$ (Use 7 intervals) [0.843]

54.3 The mid-ordinate rule

Let a required definite integral be denoted again by $\int_a^b y\,dx$ and represented by the area under the graph of $y=f(x)$ between the limits $x=a$ and $x=b$, as shown in Fig. 54.2.

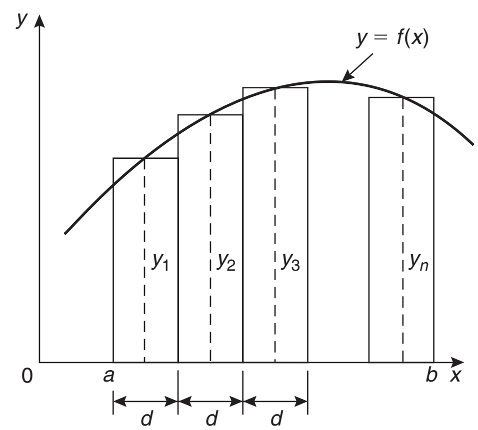

Figure 54.2

With the mid-ordinate rule each interval of width d is assumed to be replaced by a rectangle of height equal to the ordinate at the middle point of each interval, shown as $y_1, y_2, y_3, \ldots, y_n$ in Fig. 54.2.

Thus $$\int_a^b y\,dx \approx dy_1 + dy_2 + dy_3 + \cdots + dy_n \approx d(y_1+y_2+y_3+\cdots+y_n)$$

i.e. **the mid-ordinate rule states**:

$$\int_a^b y\,dx \approx \left(\text{width of interval}\right)\left(\text{sum of mid-ordinates}\right) \quad (2)$$

Problem 4. Use the mid-ordinate rule with (a) 4 intervals, (b) 8 intervals, to evaluate $\int_1^3 \frac{2}{\sqrt{x}}\,dx$, correct to 3 decimal places

(a) With 4 intervals, each will have a width of $\frac{3-1}{4}$, i.e. 0.5. and the ordinates will occur at 1.0, 1.5, 2.0, 2.5 and 3.0. Hence the mid-ordinates y_1, y_2, y_3 and y_4 occur at 1.25, 1.75, 2.25 and 2.75

Corresponding values of $\frac{2}{\sqrt{x}}$ are shown in the following table:

x	$\frac{2}{\sqrt{x}}$
1.25	1.7889
1.75	1.5119
2.25	1.3333
2.75	1.2060

From equation (2):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx (0.5)[1.7889 + 1.5119 + 1.3333 + 1.2060]$$

$$= \mathbf{2.920}, \text{ correct to 3 decimal places}$$

(b) With 8 intervals, each will have a width of 0.25 and the ordinates will occur at 1.00, 1.25, 1.50, 1.75, ... and thus mid-ordinates at 1.125, 1.375, 1.625, 1.875.... Corresponding values of $\frac{2}{\sqrt{x}}$ are shown in the following table:

x	$\frac{2}{\sqrt{x}}$
1.125	1.8856
1.375	1.7056
1.625	1.5689
1.875	1.4606
2.125	1.3720
2.375	1.2978
2.625	1.2344
2.875	1.1795

From equation (2):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx (0.25)[1.8856 + 1.7056 + 1.5689 + 1.4606 + 1.3720 + 1.2978 + 1.2344 + 1.1795]$$

$$= \mathbf{2.926}, \text{ correct to 3 decimal places}$$

As previously, the greater the number of intervals the nearer the result is to the true value of 2.928, correct to 3 decimal places.

Problem 5. Evaluate $\int_0^{2.4} e^{-x^2/3}\,dx$, correct to 4 significant figures, using the mid-ordinate rule with 6 intervals

With 6 intervals each will have a width of $\frac{2.4-0}{6}$, i.e. 0.40 and the ordinates will occur at 0, 0.40, 0.80, 1.20, 1.60, 2.00 and 2.40 and thus mid-ordinates at 0.20, 0.60, 1.00, 1.40, 1.80 and 2.20.

Corresponding values of $e^{-x^2/3}$ are shown in the following table:

x	$e^{-\frac{x^2}{3}}$
0.20	0.98676
0.60	0.88692
1.00	0.71653
1.40	0.52031
1.80	0.33960
2.20	0.19922

From equation (2):

$$\int_0^{2.4} e^{-\frac{x^2}{3}}\,dx \approx (0.40)[0.98676 + 0.88692 + 0.71653 + 0.52031 + 0.33960 + 0.19922]$$

$$= \mathbf{1.460}, \text{ correct to 4 significant figures.}$$

Now try the following exercise

Exercise 189 Further problems on the mid-ordinate rule

Evaluate the following definite integrals using the **mid-ordinate rule**, giving the answers correct to 3 decimal places.

1. $\int_0^2 \frac{3}{1+t^2}\,dt$ (Use 8 intervals) [3.323]

2. $\int_0^{\pi/2} \frac{1}{1+\sin\theta}\,d\theta$ (Use 6 intervals) [0.997]

3. $\int_1^3 \frac{\ln x}{x}\,dx$ (Use 10 intervals) [0.605]

4. $\int_0^{\pi/3} \sqrt{\cos^3 x}\,dx$ (Use 6 intervals) [0.799]

54.4 Simpson's rule

The approximation made with the trapezoidal rule is to join the top of two successive ordinates by a straight line, i.e. by using a linear approximation of the form $a+bx$. With Simpson's rule, the approximation made is to join the tops of three successive ordinates by a parabola, i.e. by using a quadratic approximation of the form $a+bx+cx^2$.

Figure 54.3 shows a parabola $y=a+bx+cx^2$ with ordinates y_1, y_2 and y_3 at $x=-d$, $x=0$ and $x=d$ respectively.

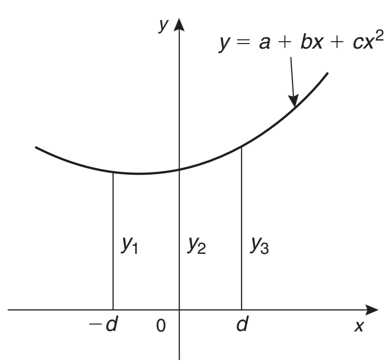

Figure 54.3

Thus the width of each of the two intervals is d. The area enclosed by the parabola, the x-axis and ordinates $x=-d$ and $x=d$ is given by:

$$\int_{-d}^{d}(a+bx+cx^2)\,dx = \left[ax+\frac{bx^2}{2}+\frac{cx^3}{3}\right]_{-d}^{d}$$

$$= \left(ad+\frac{bd^2}{2}+\frac{cd^3}{3}\right) - \left(-ad+\frac{bd^2}{2}-\frac{cd^3}{3}\right)$$

$$= 2ad+\frac{2}{3}cd^3$$

$$\text{or } \frac{1}{3}d(6a+2cd^2) \qquad (3)$$

Since $y=a+bx+cx^2$

at $x=-d,\ y_1=a-bd+cd^2$

at $x=0,\ y_2=a$

and at $x=d,\ y_3=a+bd+cd^2$

Hence $y_1+y_3=2a+2cd^2$

and $y_1+4y_2+y_3=6a+2cd^2$ (4)

Thus the area under the parabola between $x=-d$ and $x=d$ in Fig. 54.3 may be expressed as $\frac{1}{3}d(y_1+4y_2+y_3)$, from equation (3) and (4), and the result is seen to be independent of the position of the origin.

Let a definite integral be denoted by $\int_a^b y\,dx$ and represented by the area under the graph of $y=f(x)$ between the limits $x=a$ and $x=b$, as shown in Fig. 54.4. The range of integration, $b-a$, is divided into an **even** number of intervals, say $2n$, each of width d.

Since an even number of intervals is specified, an odd number of ordinates, $2n+1$, exists. Let an approximation to the curve over the first two intervals be a parabola of the form $y=a+bx+cx^2$ which passes through the tops of the three ordinates y_1, y_2 and y_3. Similarly, let an approximation to the curve over the next two intervals be the parabola which passes through the tops of the ordinates y_3, y_4 and y_5, and so on. Then

$$\int_a^b y\,dx \approx \frac{1}{3}d(y_1+4y_2+y_3)+\frac{1}{3}d(y_3+4y_4+y_5) + \frac{1}{3}d(y_{2n-1}+4y_{2n}+y_{2n+1})$$

Section 9

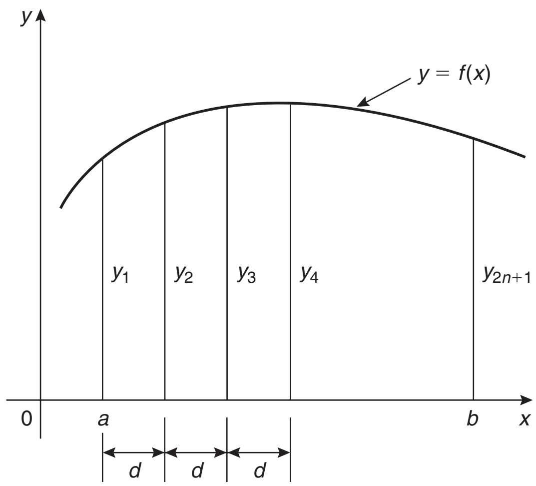

Figure 54.4

$$\approx \frac{1}{3}d[(y_1 + y_{2n+1}) + 4(y_2 + y_4 + \cdots + y_{2n}) + 2(y_3 + y_5 + \cdots + y_{2n-1})]$$

i.e. **Simpson's rule states:**

$$\int_a^b y\,dx \approx \frac{1}{3}\begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\left\{\begin{pmatrix}\text{first + last}\\ \text{ordinate}\end{pmatrix} + 4\begin{pmatrix}\text{sum of even}\\ \text{ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{sum of remaining}\\ \text{odd ordinates}\end{pmatrix}\right\} \quad (5)$$

Note that Simpson's rule can only be applied when an **even** number of intervals is chosen, i.e. an odd number of ordinates.

Problem 6. Use Simpson's rule with (a) 4 intervals, (b) 8 intervals, to evaluate $\int_1^3 \frac{2}{\sqrt{x}}\,dx$, correct to 3 decimal places

(a) With 4 intervals, each will have a width of $\frac{3-1}{4}$ i.e. 0.5 and the ordinates will occur at 1.0, 1.5, 2.0, 2.5 and 3.0.

The values of the ordinates are as shown in the table of Problem 1(b), page 470.

Thus, from equation (5):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx \frac{1}{3}(0.5)[(2.0000 + 1.1547) + 4(1.6330 + 1.2649) + 2(1.4142)]$$
$$= \frac{1}{3}(0.5)[3.1547 + 11.5916 + 2.8284]$$

$= \mathbf{2.929}$, correct to 3 decimal places.

(b) With 8 intervals, each will have a width of $\frac{3-1}{8}$ i.e. 0.25 and the ordinates occur at 1.00, 1.25, 1.50, 1.75, …, 3.0.

The values of the ordinates are as shown in the table in Problem 2, page 470.

Thus, from equation (5):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx \frac{1}{3}(0.25)[(2.0000 + 1.1547) + 4(1.7889 + 1.5119 + 1.3333 + 1.2060) + 2(1.6330 + 1.4142 + 1.2649)]$$
$$= \frac{1}{3}(0.25)[3.1547 + 23.3604 + 8.6242]$$

$= \mathbf{2.928}$, correct to 3 decimal places.

It is noted that the latter answer is exactly the same as that obtained by integration. In general, Simpson's rule is regarded as the most accurate of the three approximate methods used in numerical integration.

Problem 7. Evaluate $\int_0^{\pi/3}\sqrt{1 - \frac{1}{3}\sin^2\theta}\,d\theta$, correct to 3 decimal places, using Simpson's rule with 6 intervals

With 6 intervals, each will have a width of $\frac{\frac{\pi}{3}-0}{6}$ i.e. $\frac{\pi}{18}$ rad (or 10°), and the ordinates will occur at 0, $\frac{\pi}{18}, \frac{\pi}{9}, \frac{\pi}{6}, \frac{2\pi}{9}, \frac{5\pi}{18}$ and $\frac{\pi}{3}$

Section 9

Corresponding values of $\sqrt{1-\frac{1}{3}\sin^2\theta}$ are shown in the table below:

θ	0	$\frac{\pi}{18}$ (or 10°)	$\frac{\pi}{9}$ (or 20°)	$\frac{\pi}{6}$ (or 30°)
$\sqrt{1-\frac{1}{3}\sin^2\theta}$	1.0000	0.9950	0.9803	0.9574

θ	$\frac{2\pi}{9}$ (or 40°)	$\frac{5\pi}{18}$ (or 50°)	$\frac{\pi}{3}$ (or 60°)
$\sqrt{1-\frac{1}{3}\sin^2\theta}$	0.9286	0.8969	0.8660

From equation (5):

$$\int_0^{\frac{\pi}{3}} \sqrt{1-\frac{1}{3}\sin^2\theta}\, d\theta$$

$$\approx \frac{1}{3}\left(\frac{\pi}{18}\right)[(1.0000+0.8660)+4(0.9950+0.9574+0.8969)+2(0.9803+0.9286)]$$

$$= \frac{1}{3}\left(\frac{\pi}{18}\right)[1.8660+11.3972+3.8178]$$

$= \mathbf{0.994}$, correct to 3 decimal places.

Problem 8. An alternating current i has the following values at equal intervals of 2.0 milliseconds:

Time (ms)	0	2.0	4.0	6.0	8.0	10.0	12.0
Current i (A)	0	3.5	8.2	10.0	7.3	2.0	0

Charge, q, in millicoulombs, is given by $q=\int_0^{12.0} i\,dt$. Use Simpson's rule to determine the approximate charge in the 12 ms period

From equation (5):

$$\text{Charge,}\quad q=\int_0^{12.0} i\,dt$$

$$\approx \frac{1}{3}(2.0)[(0+0)+4(3.5+10.0+2.0)+2(8.2+7.3)]$$

$= \mathbf{62\ mC}$

Now try the following exercise

Exercise 190 Further problems on Simpson's rule

In Problems 1 to 5, evaluate the definite integrals using **Simpson's rule**, giving the answers correct to 3 decimal places.

1. $\int_0^{\pi/2} \sqrt{\sin x}\, dx$ (Use 6 intervals) [1.187]
2. $\int_0^{1.6} \frac{1}{1+\theta^4}\, d\theta$ (Use 8 intervals) [1.034]
3. $\int_{0.2}^{1.0} \frac{\sin\theta}{\theta}\, d\theta$ (Use 8 intervals) [0.747]
4. $\int_0^{\pi/2} x\cos x\, dx$ (Use 6 intervals) [0.571]
5. $\int_0^{\pi/3} e^{x^2}\sin 2x\, dx$ (Use 10 intervals) [1.260]

In Problems 6 and 7 evaluate the definite integrals using (a) integration, (b) the trapezoidal rule, (c) the mid-ordinate rule, (d) Simpson's rule. Give answers correct to 3 decimal places.

6. $\int_1^4 \frac{4}{x^3}\, dx$ (Use 6 intervals) [(a) 1.875 (b) 2.107 (c) 1.765 (d) 1.916]
7. $\int_2^6 \frac{1}{\sqrt{2x-1}}\, dx$ (Use 8 intervals) [(a) 1.585 (b) 1.588 (c) 1.583 (d) 1.585]

In Problems 8 and 9 evaluate the definite integrals using (a) the trapezoidal rule, (b) the mid-ordinate rule, (c) Simpson's rule. Use 6 intervals in each case and give answers correct to 3 decimal places.

8. $\int_0^3 \sqrt{1+x^4}\, dx$ [(a) 10.194 (b) 10.007 (c) 10.070]
9. $\int_{0.1}^{0.7} \frac{1}{\sqrt{1-y^2}}\, dy$ [(a) 0.677 (b) 0.674 (c) 0.675]

10. A vehicle starts from rest and its velocity is measured every second for 8 seconds, with values as follows:

time t (s)	velocity v (ms^{-1})
0	0
1.0	0.4
2.0	1.0
3.0	1.7
4.0	2.9
5.0	4.1
6.0	6.2
7.0	8.0
8.0	9.4

The distance travelled in 8.0 seconds is given by $\int_0^{8.0} v\,dt$.

Estimate this distance using Simpson's rule, giving the answer correct to 3 significant figures. [28.8 m]

11. A pin moves along a straight guide so that its velocity v (m/s) when it is a distance x (m) from the beginning of the guide at time t (s) is given in the table below:

t (s)	v (m/s)
0	0
0.5	0.052
1.0	0.082
1.5	0.125
2.0	0.162
2.5	0.175
3.0	0.186
3.5	0.160
4.0	0

Use Simpson's rule with 8 intervals to determine the approximate total distance travelled by the pin in the 4.0 second period. [0.485 m]

Revision Test 15

This Revision test covers the material contained in Chapters 51 to 54. *The marks for each question are shown in brackets at the end of each question.*

1. Determine: (a) $\displaystyle\int \frac{x-11}{x^2-x-2}\,dx$

   (b) $\displaystyle\int \frac{3-x}{(x^2+3)(x+3)}dx$ (21)

2. Evaluate: $\displaystyle\int_1^2 \frac{3}{x^2(x+2)}\,dx$ correct to 4 significant figures. (11)

3. Determine: $\displaystyle\int \frac{dx}{2\sin x + \cos x}$ (7)

4. Determine the following integrals:

   (a) $\displaystyle\int 5xe^{2x}dx$ (b) $\displaystyle\int t^2 \sin 2t\, dt$ (12)

5. Evaluate correct to 3 decimal places:

   $\displaystyle\int_1^4 \sqrt{x}\ln x\, dx$ (9)

6. Evaluate: $\displaystyle\int_1^3 \frac{5}{x^2}\,dx$ using

   (a) integration

   (b) the trapezoidal rule

   (c) the mid-ordinate rule

   (d) Simpson's rule.

   In each of the approximate methods use 8 intervals and give the answers correct to 3 decimal places. (16)

7. An alternating current i has the following values at equal intervals of 5 ms:

Time t (ms)	0	5	10	15	20	25	30
Current i (A)	0	4.8	9.1	12.7	8.8	3.5	0

   Charge q, in coulombs, is given by

   $$q = \int_0^{30\times 10^{-3}} i\, dt.$$

   Use Simpson's rule to determine the approximate charge in the 30 ms period. (4)

Chapter 55

Areas under and between curves

55.1 Area under a curve

The area shown shaded in Fig. 55.1 may be determined using approximate methods (such as the trapezoidal rule, the mid-ordinate rule or Simpson's rule) or, more precisely, by using integration.

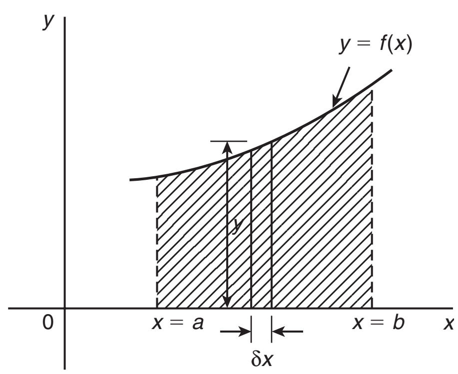

Figure 55.1

(i) Let A be the area shown shaded in Fig. 55.1 and let this area be divided into a number of strips each of width δx. One such strip is shown and let the area of this strip be δA.

Then: $\delta A \approx y\delta x$ (1)

The accuracy of statement (1) increases when the width of each strip is reduced, i.e. area A is divided into a greater number of strips.

(ii) Area A is equal to the sum of all the strips from $x=a$ to $x=b$,

i.e. $A = \underset{\delta x \to 0}{\text{limit}} \sum_{x=a}^{x=b} y\,\delta x$ (2)

(iii) From statement (1), $\dfrac{\delta A}{\delta x} \approx y$ (3)

In the limit, as δx approaches zero, $\dfrac{\delta A}{\delta x}$ becomes the differential coefficient $\dfrac{dA}{dx}$

Hence $\underset{\delta x \to 0}{\text{limit}} \left(\dfrac{\delta A}{\delta x}\right) = \dfrac{dA}{dx} = y$, from statement (3).

By integration,

$$\int \frac{dA}{dx}dx = \int y\,dx \quad \text{i.e.} \quad A = \int y\,dx$$

The ordinates $x=a$ and $x=b$ limit the area and such ordinate values are shown as limits. Hence

$$A = \int_a^b y\,dx \qquad (4)$$

(iv) Equating statements (2) and (4) gives:

$$\textbf{Area } A = \underset{\delta x \to 0}{\textbf{limit}} \sum_{x=a}^{x=b} y\,\delta x = \int_a^b y\,dx$$

$$= \int_a^b f(x)\,dx$$

(v) If the area between a curve $x=f(y)$, the y-axis and ordinates $y=p$ and $y=q$ is required, then

$$\text{area} = \int_p^q x\,dy$$

Thus, determining the area under a curve by integration merely involves evaluating a definite integral.

There are several instances in engineering and science where the area beneath a curve needs to be accurately determined. For example, the areas between limits of a:

velocity/time graph gives distance travelled,
force/distance graph gives work done,
voltage/current graph gives power, and so on.

Should a curve drop below the x-axis, then $y\ (=f(x))$ becomes negative and $f(x)\,dx$ is negative. When determining such areas by integration, a negative sign is placed before the integral. For the curve shown in Fig. 55.2, the total shaded area is given by (area E + area F + area G).

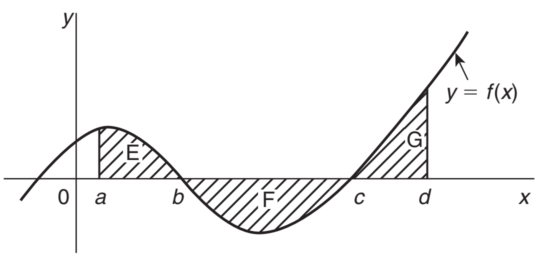

Figure 55.2

By integration, **total shaded area**

$$= \int_a^b f(x)\,dx - \int_b^c f(x)\,dx + \int_c^d f(x)\,dx$$

(Note that this is **not** the same as $\int_a^d f(x)\,dx$.)
It is usually necessary to sketch a curve in order to check whether it crosses the x-axis.

55.2 Worked problems on the area under a curve

Problem 1. Determine the area enclosed by $y=2x+3$, the x-axis and ordinates $x=1$ and $x=4$

$y=2x+3$ is a straight line graph as shown in Fig. 55.3, where the required area is shown shaded.
By integration,

$$\text{shaded area} = \int_1^4 y\,dx = \int_1^4 (2x+3)\,dx = \left[\frac{2x^2}{2}+3x\right]_1^4 = [(16+12)-(1+3)] = \textbf{24 square units}$$

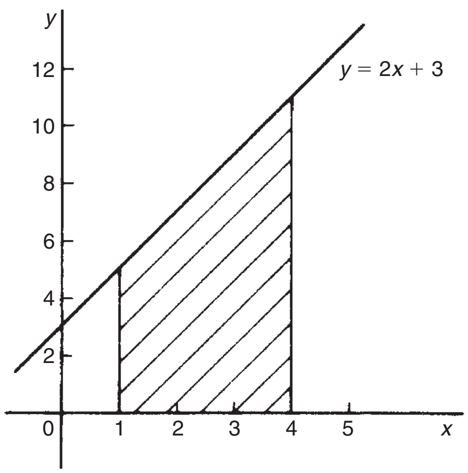

Figure 55.3

[This answer may be checked since the shaded area is a trapezium.

Area of trapezium

$$= \frac{1}{2}\begin{pmatrix}\text{sum of parallel}\\ \text{sides}\end{pmatrix}\begin{pmatrix}\text{perpendicular distance}\\ \text{between parallel sides}\end{pmatrix}$$

$$= \frac{1}{2}(5+11)(3)$$

= **24 square units**]

Problem 2. The velocity v of a body t seconds after a certain instant is: $(2t^2+5)$ m/s. Find by integration how far it moves in the interval from $t=0$ to $t=4$ s

Since $2t^2+5$ is a quadratic expression, the curve $v=2t^2+5$ is a parabola cutting the v-axis at $v=5$, as shown in Fig. 55.4.
The distance travelled is given by the area under the v/t curve (shown shaded in Fig. 55.4).
By integration,

$$\text{shaded area} = \int_0^4 v\,dt = \int_0^4 (2t^2+5)\,dt = \left[\frac{2t^3}{3}+5t\right]_0^4 = \left(\frac{2(4^3)}{3}+5(4)\right)-(0)$$

i.e. **distance travelled = 62.67 m**

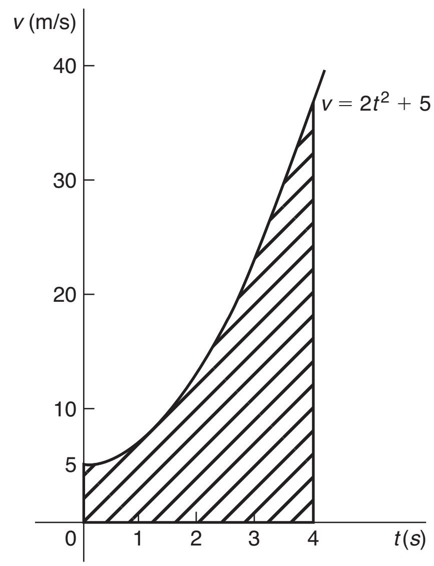

Figure 55.4

Problem 3. Sketch the graph $y = x^3 + 2x^2 - 5x - 6$ between $x = -3$ and $x = 2$ and determine the area enclosed by the curve and the x-axis

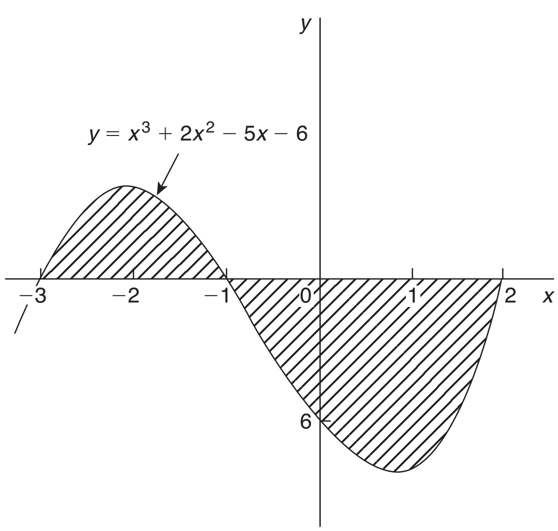

Figure 55.5

A table of values is produced and the graph sketched as shown in Fig. 55.5 where the area enclosed by the curve and the x-axis is shown shaded.

x	−3	−2	−1	0	1	2
x^3	−27	−8	−1	0	1	8
$2x^2$	18	8	2	0	2	8
$-5x$	15	10	5	0	−5	−10
−6	−6	−6	−6	−6	−6	−6
y	0	4	0	−6	−8	0

Shaded area $= \int_{-3}^{-1} y\,dx - \int_{-1}^{2} y\,dx$, the minus sign before the second integral being necessary since the enclosed area is below the x-axis.

Hence shaded area

$$= \int_{-3}^{-1} (x^3 + 2x^2 - 5x - 6)\,dx - \int_{-1}^{2} (x^3 + 2x^2 - 5x - 6)\,dx$$

$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{5x^2}{2} - 6x\right]_{-3}^{-1} - \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{5x^2}{2} - 6x\right]_{-1}^{2}$$

$$= \left[\left\{\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6\right\} - \left\{\frac{81}{4} - 18 - \frac{45}{2} + 18\right\}\right] - \left[\left\{4 + \frac{16}{3} - 10 - 12\right\} - \left\{\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6\right\}\right]$$

$$= \left[\left\{3\frac{1}{12}\right\} - \left\{-2\frac{1}{4}\right\}\right] - \left[\left\{-12\frac{2}{3}\right\} - \left\{3\frac{1}{12}\right\}\right]$$

$$= \left[5\frac{1}{3}\right] - \left[-15\frac{3}{4}\right]$$

$$= \mathbf{21\frac{1}{12}} \quad \text{or} \quad \textbf{21.08 square units}$$

Problem 4. Determine the area enclosed by the curve $y = 3x^2 + 4$, the x-axis and ordinates $x = 1$ and $x = 4$ by (a) the trapezoidal rule, (b) the mid-ordinate rule, (c) Simpson's rule, and (d) integration

x	0	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	4	7	10.75	16	22.75	31	40.75	52

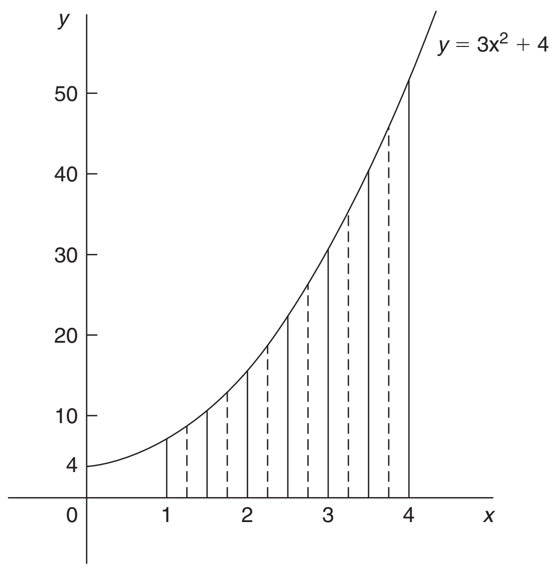

Figure 55.6

The curve $y = 3x^2 + 4$ is shown plotted in Fig. 55.6.

(a) **By the trapezoidal rule**

$$\text{Area} = \begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\left[\frac{1}{2}\begin{pmatrix}\text{first + last}\\ \text{ordinate}\end{pmatrix} + \begin{pmatrix}\text{sum of}\\ \text{remaining}\\ \text{ordinates}\end{pmatrix}\right]$$

Selecting 6 intervals each of width 0.5 gives:

$$\text{Area} = (0.5)\left[\frac{1}{2}(7+52) + 10.75 + 16 + 22.75 + 31 + 40.75\right]$$

$$= \textbf{75.375 square units}$$

(b) **By the mid-ordinate rule**, area = (width of interval) (sum of mid-ordinates). Selecting 6 intervals, each of width 0.5 gives the mid-ordinates as shown by the broken lines in Fig. 55.6.

$$\text{Thus, area} = (0.5)(8.5 + 13 + 19 + 26.5 + 35.5 + 46)$$

$$= \textbf{74.25 square units}$$

(c) **By Simpson's rule**,

$$\text{area} = \frac{1}{3}\begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\left[\begin{pmatrix}\text{first + last}\\ \text{ordinates}\end{pmatrix} + 4\begin{pmatrix}\text{sum of even}\\ \text{ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{sum of remaining}\\ \text{odd ordinates}\end{pmatrix}\right]$$

Selecting 6 intervals, each of width 0.5, gives:

$$\text{area} = \frac{1}{3}(0.5)[(7+52) + 4(10.75 + 22.75 + 40.75) + 2(16+31)]$$

$$= \textbf{75 square units}$$

(d) **By integration**, shaded area

$$= \int_1^4 y\,dx$$

$$= \int_1^4 (3x^2 + 4)\,dx$$

$$= \left[x^3 + 4x\right]_1^4$$

$$= \textbf{75 square units}$$

Integration gives the precise value for the area under a curve. In this case Simpson's rule is seen to be the most accurate of the three approximate methods.

Problem 5. Find the area enclosed by the curve $y = \sin 2x$, the x-axis and the ordinates $x = 0$ and $x = \pi/3$

A sketch of $y = \sin 2x$ is shown in Fig. 55.7.

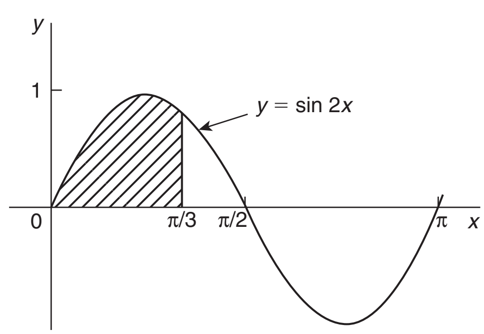

Figure 55.7

(Note that $y = \sin 2x$ has a period of $\frac{2\pi}{2}$, i.e. π radians.)

$$\text{Shaded area} = \int_0^{\pi/3} y\,dx$$

$$= \int_0^{\pi/3} \sin 2x\,dx$$

$$= \left[-\frac{1}{2}\cos 2x\right]_0^{\pi/3}$$

$$= \left\{-\frac{1}{2}\cos\frac{2\pi}{3}\right\} - \left\{-\frac{1}{2}\cos 0\right\}$$

$$= \left\{-\frac{1}{2}\left(-\frac{1}{2}\right)\right\} - \left\{-\frac{1}{2}(1)\right\}$$

$$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \text{ square units}$$

Now try the following exercise

Exercise 191 Further problems on area under curves

Unless otherwise stated all answers are in square units.

1. Shown by integration that the area of the triangle formed by the line $y = 2x$, the ordinates $x = 0$ and $x = 4$ and the x-axis is 16 square units.

2. Sketch the curve $y = 3x^2 + 1$ between $x = -2$ and $x = 4$. Determine by integration the area enclosed by the curve, the x-axis and ordinates $x = -1$ and $x = 3$. Use an approximate method to find the area and compare your result with that obtained by integration. [32]

In Problems 3 to 8, find the area enclosed between the given curves, the horizontal axis and the given ordinates.

3. $y = 5x$; $x = 1, x = 4$ [37.5]

4. $y = 2x^2 - x + 1$; $x = -1, x = 2$ [7.5]

5. $y = 2\sin 2\theta$; $\theta = 0, \theta = \frac{\pi}{4}$ [1]

6. $\theta = t + e^t$; $t = 0, t = 2$ [8.389]

7. $y = 5\cos 3t$; $t = 0, t = \frac{\pi}{6}$ [1.67]

8. $y = (x - 1)(x - 3)$; $x = 0, x = 3$ [2.67]

55.3 Further worked problems on the area under a curve

Problem 6. A gas expands according to the law $pv = \text{constant}$. When the volume is $3\,\text{m}^3$ the pressure is 150 kPa. Given that work done $= \int_{v_1}^{v_2} p\,dv$, determine the work done as the gas expands from $2\,\text{m}^3$ to a volume of $6\,\text{m}^3$

$pv = \text{constant}$. When $v = 3\,\text{m}^3$ and $p = 150\,\text{kPa}$ the constant is given by $(3 \times 150) = 450\,\text{kPa m}^3$ or 450 kJ.

Hence $pv = 450$, or $p = \dfrac{450}{v}$

$$\text{Work done} = \int_2^6 \frac{450}{v}\,dv$$

$$= [450 \ln v]_2^6 = 450[\ln 6 - \ln 2]$$

$$= 450 \ln \frac{6}{2} = 450 \ln 3 = \mathbf{494.4\,kJ}$$

Problem 7. Determine the area enclosed by the curve $y = 4\cos\left(\frac{\theta}{2}\right)$, the θ-axis and ordinates $\theta = 0$ and $\theta = \frac{\pi}{2}$

The curve $y = 4\cos(\theta/2)$ is shown in Fig. 55.8.

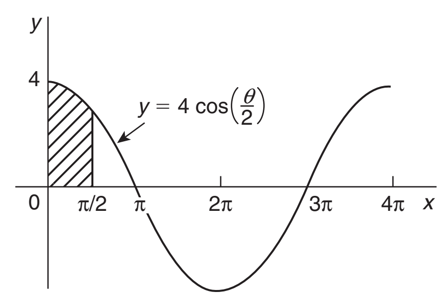

Figure 55.8

(Note that $y = 4\cos\left(\frac{\theta}{2}\right)$ has a maximum value of 4 and period $2\pi/(1/2)$, i.e. 4π rads.)

$$\text{Shaded area} = \int_0^{\pi/2} y\,d\theta = \int_0^{\pi/2} 4\cos\frac{\theta}{2}\,d\theta$$

$$= \left[4\left(\frac{1}{\frac{1}{2}}\right)\sin\frac{\theta}{2}\right]_0^{\pi/2}$$

$$= \left(8\sin\frac{\pi}{4}\right) - (8\sin 0)$$

$$= \mathbf{5.657\ square\ units}$$

Problem 8. Determine the area bounded by the curve $y = 3e^{t/4}$, the t-axis and ordinates $t = -1$ and $t = 4$, correct to 4 significant figures

A table of values is produced as shown.

t	-1	0	1	2	3	4
$y = 3e^{t/4}$	2.34	3.0	3.85	4.95	6.35	8.15

Since all the values of y are positive the area required is wholly above the t-axis.

$$\text{Hence area} = \int_1^4 y\,dt$$

$$= \int_1^4 3e^{t/4}dt = \left[\frac{3}{\left(\frac{1}{4}\right)}e^{t/4}\right]_{-1}^4$$

$$= 12\left[e^{t/4}\right]_{-1}^4 = 12(e^1 - e^{-1/4})$$

$$= 12(2.7183 - 0.7788)$$

$$= 12(1.9395) = \mathbf{23.27\ square\ units}$$

Problem 9. Sketch the curve $y = x^2 + 5$ between $x = -1$ and $x = 4$. Find the area enclosed by the curve, the x-axis and the ordinates $x = 0$ and $x = 3$. Determine also, by integration, the area enclosed by the curve and the y-axis, between the same limits

A table of values is produced and the curve $y = x^2 + 5$ plotted as shown in Fig. 55.9.

x	-1	0	1	2	3
y	6	5	6	9	14

$$\text{Shaded area} = \int_0^3 y\,dx = \int_0^3 (x^2 + 5)\,dx$$

$$= \left[\frac{x^3}{5} + 5x\right]_0^3$$

$$= \mathbf{24\ square\ units}$$

When $x = 3$, $y = 3^2 + 5 = 14$, and when $x = 0$, $y = 5$.

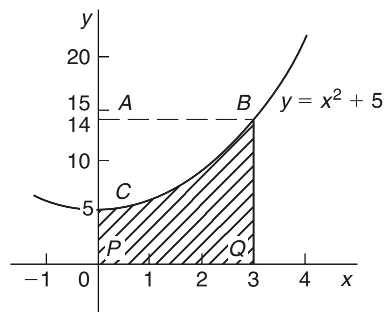

Figure 55.9

Since $y = x^2 + 5$ then $x^2 = y - 5$ and $x = \sqrt{y-5}$
The area enclosed by the curve $y = x^2 + 5$ (i.e. $x = \sqrt{y-5}$), the y-axis and the ordinates $y = 5$ and $y = 14$ (i.e. area ABC of Fig. 55.9) is given by:

$$\text{Area} = \int_{y=5}^{y=14} x\,dy = \int_5^{14} \sqrt{y-5}\,dy$$

$$= \int_5^{14} (y-5)^{1/2}\,dy$$

Let $u = y - 5$, then $\frac{du}{dy} = 1$ and $dy = du$

Hence $\int (y-5)^{1/2}dy = \int u^{1/2}du = \frac{2}{3}u^{3/2}$
(for algebraic substitutions, see Chapter 49)
Since $u = y - 5$ then

$$\int_5^{14} \sqrt{y-5}\,dy = \frac{2}{3}\left[(y-5)^{3/2}\right]_5^{14}$$

$$= \frac{2}{3}\left[\sqrt{9^3} - 0\right]$$

$$= \mathbf{18\ square\ units}$$

(Check: From Fig. 55.9, area BCPQ + area $ABC = 24 + 18 = 42$ square units, which is the area of rectangle $ABQP$.)

Problem 10. Determine the area between the curve $y = x^3 - 2x^2 - 8x$ and the x-axis

$$y = x^3 - 2x^2 - 8x = x(x^2 - 2x - 8)$$

$$= x(x+2)(x-4)$$

When $y = 0$, then $x = 0$ or $(x+2) = 0$ or $(x-4) = 0$, i.e. when $y = 0$, $x = 0$ or -2 or 4, which means that the curve crosses the x-axis at 0, -2 and 4. Since the curve is a continuous function, only one other co-ordinate value needs to be calculated before a sketch of the curve can be produced. When $x=1$, $y=-9$, showing that the part of the curve between $x=0$ and $x=4$ is negative. A sketch of $y=x^3-2x^2-8x$ is shown in Fig. 55.10. (Another method of sketching Fig. 55.10 would have been to draw up a table of values.)

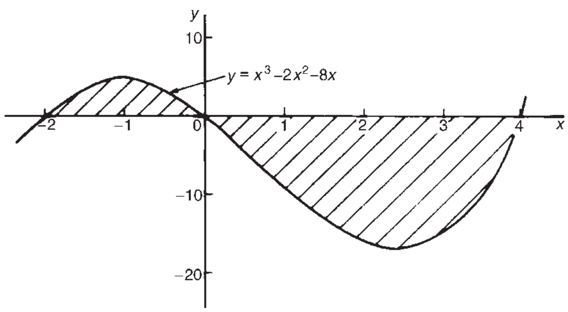

Figure 55.10

$$\text{Shaded area} = \int_{-2}^{0}(x^3-2x^2-8x)\,dx - \int_{0}^{4}(x^3-2x^2-8x)\,dx$$

$$= \left[\frac{x^4}{4}-\frac{2x^3}{3}-\frac{8x^2}{2}\right]_{-2}^{0} - \left[\frac{x^4}{4}-\frac{2x^3}{3}-\frac{8x^2}{2}\right]_{0}^{4}$$

$$= \left(6\frac{2}{3}\right)-\left(-42\frac{2}{3}\right)$$

$$= \mathbf{49\frac{1}{3}\ square\ units}$$

Now try the following exercise

Exercise 192 Further problems on areas under curves

In Problems 1 and 2, find the area enclosed between the given curves, the horizontal axis and the given ordinates.

1. $y=2x^3$; $x=-2, x=2$ [16 square units]
2. $xy=4$; $x=1, x=4$ [5.545 square units]
3. The force F newtons acting on a body at a distance x metres from a fixed point is given by: $F=3x+2x^2$. If work done $=\int_{x_1}^{x_2}F\,dx$, determine the work done when the body moves from the position where $x=1$ m to that where $x=3$ m. [29.33 Nm]
4. Find the area between the curve $y=4x-x^2$ and the x-axis. [10.67 square units]
5. Determine the area enclosed by the curve $y=5x^2+2$, the x-axis and the ordinates $x=0$ and $x=3$. Find also the area enclosed by the curve and the y-axis between the same limits. [51 sq. units, 90 sq. units]
6. Calculate the area enclosed between $y=x^3-4x^2-5x$ and the x-axis. [73.83 sq. units]
7. The velocity v of a vehicle t seconds after a certain instant is given by: $v=(3t^2+4)$ m/s. Determine how far it moves in the interval from $t=1$ s to $t=5$ s. [140 m]
8. A gas expands according to the law $pv=$ constant. When the volume is 2 m^3 the pressure is 250 kPa. Find the work done as the gas expands from 1 m^3 to a volume of 4 m^3 given that work done $=\int_{v_1}^{v_2}p\,dv$ [693.1 kJ]

55.4 The area between curves

The area enclosed between curves $y=f_1(x)$ and $y=f_2(x)$ (shown shaded in Fig. 55.11) is given by:

$$\text{shaded area} = \int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx$$

$$= \mathbf{\int_a^b [f_2(x)-f_2(x)]\,dx}$$

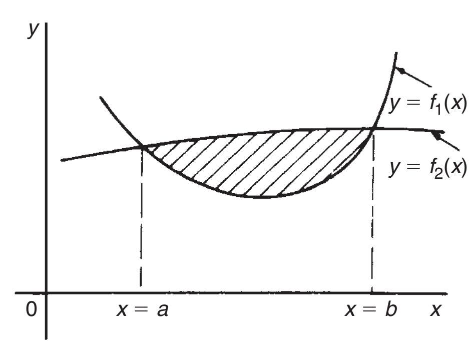

Figure 55.11

Problem 11. Determine the area enclosed between the curves $y=x^2+1$ and $y=7-x$

At the points of intersection, the curves are equal. Thus, equating the y-values of each curve gives: $x^2+1=7-x$, from which $x^2+x-6=0$. Factorising gives $(x-2)(x+3)=0$, from which, $x=2$ and $x=-3$. By firstly determining the points of intersection the range of x-values has been found. Tables of values are produced as shown below.

x	-3	-2	-1	0	1	2
$y=x^2+1$	10	5	2	1	2	5

x	-3	0	2
$y=7-x$	10	7	5

A sketch of the two curves is shown in Fig. 55.12.

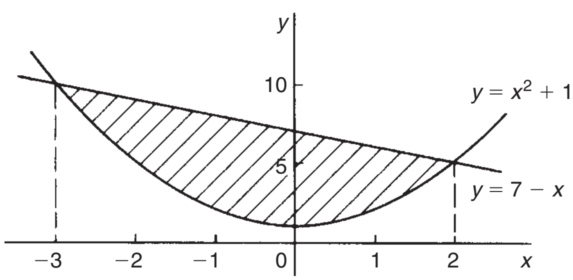

Figure 55.12

$$\text{Shaded area} = \int_{-3}^{2}(7-x)dx - \int_{-3}^{2}(x^2+1)dx$$
$$= \int_{-3}^{2}[(7-x)-(x^2+1)]dx$$
$$= \int_{-3}^{2}(6-x-x^2)dx$$
$$= \left[6x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-3}^{2}$$
$$= \left(12-2-\frac{8}{3}\right)-\left(-18-\frac{9}{2}+9\right)$$
$$= \left(7\frac{1}{3}\right)-\left(-13\frac{1}{2}\right)$$
$$= \mathbf{20\frac{5}{6}\ \text{square units}}$$

Problem 12. (a) Determine the coordinates of the points of intersection of the curves $y=x^2$ and $y^2=8x$. (b) Sketch the curves $y=x^2$ and $y^2=8x$ on the same axes. (c) Calculate the area enclosed by the two curves

(a) At the points of intersection the coordinates of the curves are equal. When $y=x^2$ then $y^2=x^4$.

Hence at the points of intersection $x^4=8x$, by equating the y^2 values.

Thus $x^4-8x=0$, from which $x(x^3-8)=0$, i.e. $x=0$ or $(x^3-8)=0$.

Hence at the points of intersection $x=0$ or $x=2$.

When $x=0$, $y=0$ and when $x=2$, $y=2^2=4$.

Hence the points of intersection of the curves $y=x^2$ and $y^2=8x$ are (0, 0) and (2, 4)

(b) A sketch of $y=x^2$ and $y^2=8x$ is shown in Fig. 55.13

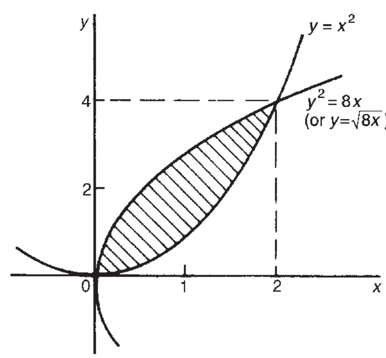

Figure 55.13

(c) **Shaded area** $= \int_0^2\{\sqrt{8x}-x^2\}dx$

$$= \int_0^2\{(\sqrt{8})x^{1/2}-x^2\}dx$$
$$= \left[(\sqrt{8})\frac{x^{3/2}}{(\frac{3}{2})}-\frac{x^3}{3}\right]_0^2$$
$$= \left\{\frac{\sqrt{8}\sqrt{8}}{(\frac{3}{2})}-\frac{8}{3}\right\}-\{0\}$$
$$= \frac{16}{3}-\frac{8}{3}=\frac{8}{3}$$
$$= \mathbf{2\frac{2}{3}\ \text{square units}}$$

Problem 13. Determine by integration the area bounded by the three straight lines $y = 4 - x$, $y = 3x$ and $3y = x$

Each of the straight lines is shown sketched in Fig. 55.14.

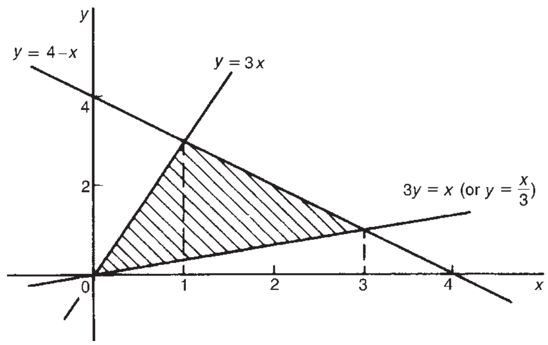

Figure 55.14

$$\text{Shaded area} = \int_0^1 \left(3x - \frac{x}{3}\right)dx + \int_1^3 \left[(4-x) - \frac{x}{3}\right]dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^2}{6}\right]_0^1 + \left[4x - \frac{x^2}{2} - \frac{x^2}{6}\right]_1^3$$

$$= \left[\left(\frac{3}{2} - \frac{1}{6}\right) - (0)\right] + \left[\left(12 - \frac{9}{2} - \frac{9}{6}\right) - \left(4 - \frac{1}{2} - \frac{1}{6}\right)\right]$$

$$= \left(1\frac{1}{3}\right) + \left(6 - 3\frac{1}{3}\right)$$

$$= \mathbf{4\ square\ units}$$

Now try the following exercise

Exercise 193 Further problems on areas between curves

1. Determine the coordinates of the points of intersection and the area enclosed between the parabolas $y^2 = 3x$ and $x^2 = 3y$.
   [(0, 0) and (3, 3), 3 sq. units]

2. Sketch the curves $y = x^2 + 3$ and $y = 7 - 3x$ and determine the area enclosed by them.
   [20.83 square units]

3. Determine the area enclosed by the curves $y = \sin x$ and $y = \cos x$ and the y-axis.
   [0.4142 square units]

4. Determine the area enclosed by the three straight lines $y = 3x$, $2y = x$ and $y + 2x = 5$
   [2.5 sq. units]

Chapter 56

Mean and root mean square values

56.1 Mean or average values

(i) The mean or average value of the curve shown in Fig. 56.1, between $x = a$ and $x = b$, is given by: **mean or average value,**

$$\bar{y} = \frac{\textbf{area under curve}}{\textbf{length of base}}$$

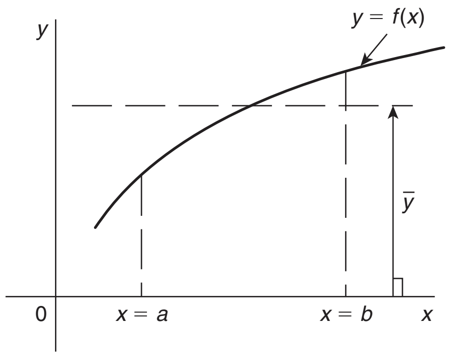

Figure 56.1

(ii) When the area under a curve may be obtained by integration then:
mean or average value,

$$\bar{y} = \frac{\int_a^b y\,dx}{b - a}$$

i.e. $$\bar{y} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

(iii) For a periodic function, such as a sine wave, the mean value is assumed to be 'the mean value over half a cycle', since the mean value over a complete cycle is zero.

Problem 1. Determine, using integration, the mean value of $y = 5x^2$ between $x = 1$ and $x = 4$

Mean value,

$$\bar{y} = \frac{1}{4-1}\int_1^4 y\,dx = \frac{1}{3}\int_1^4 5x^2\,dx$$

$$= \frac{1}{3}\left[\frac{5x^3}{3}\right]_1^4 = \frac{5}{9}[x^3]_1^4 = \frac{5}{9}(64-1) = \mathbf{35}$$

Problem 2. A sinusoidal voltage is given by $v = 100 \sin \omega t$ volts. Determine the mean value of the voltage over half a cycle using integration

Half a cycle means the limits are 0 to π radians. Mean value,

$$\bar{v} = \frac{1}{\pi - 0}\int_0^\pi v\,d(\omega t)$$

$$= \frac{1}{\pi}\int_0^\pi 100 \sin \omega t\,d(\omega t) = \frac{100}{\pi}[-\cos \omega t]_0^\pi$$

$$= \frac{100}{\pi}[(-\cos \pi) - (-\cos 0)]$$

$$= \frac{100}{\pi}[(+1) - (-1)] = \frac{200}{\pi}$$

$$= \mathbf{63.66\ volts}$$

[Note that for a sine wave,

$$\textbf{mean value} = \frac{2}{\pi} \times \textbf{maximum value}$$

In this case, mean value $= \frac{2}{\pi} \times 100 = 63.66\,\text{V}]$

Problem 3. Calculate the mean value of $y = 3x^2 + 2$ in the range $x = 0$ to $x = 3$ by (a) the mid-ordinate rule and (b) integration

(a) A graph of $y = 3x^2$ over the required range is shown in Fig. 56.2 using the following table:

x	0	0.5	1.0	1.5	2.0	2.5	3.0
y	2.0	2.75	5.0	8.75	14.0	20.75	29.0

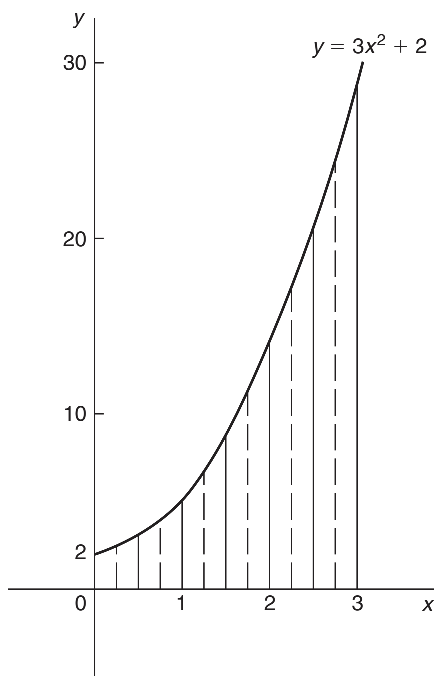

Figure 56.2

Using the mid-ordinate rule, mean value

$$= \frac{\text{area under curve}}{\text{length of base}} = \frac{\text{sum of mid-ordinates}}{\text{number of mid-ordinates}}$$

Selecting 6 intervals, each of width 0.5, the mid-ordinates are erected as shown by the broken lines in Fig. 56.2.

$$\text{Mean value} = \frac{2.2 + 3.7 + 6.7 + 11.2 + 17.2 + 24.7}{6} = \frac{65.7}{6} = \mathbf{10.95}$$

(b) By integration, mean value

$$= \frac{1}{3-0}\int_0^3 y\,dx = \frac{1}{3}\int_0^3 (3x^2 + 2)dx$$

$$= \frac{1}{3}[x^3 + 2x]_0^3 = \frac{1}{3}[(27 + 6) - (0)]$$

$$= \mathbf{11}$$

The answer obtained by integration is exact; greater accuracy may be obtained by the mid-ordinate rule if a larger number of intervals are selected.

Problem 4. The number of atoms, N, remaining in a mass of material during radioactive decay after time t seconds is given by: $N = N_0 e^{-\lambda t}$, where N_0 and λ are constants. Determine the mean number of atoms in the mass of material for the time period $t = 0$ and $t = \frac{1}{\lambda}$

Mean number of atoms

$$= \frac{1}{\frac{1}{\lambda} - 0}\int_0^{1/\lambda} N\,dt = \frac{1}{\frac{1}{\lambda}}\int_0^{1/\lambda} N_0 e^{-\lambda t}\,dt$$

$$= \lambda N_0 \int_0^{1/\lambda} e^{-\lambda t}\,dt = \lambda N_0 \left[\frac{e^{-\lambda t}}{-\lambda}\right]_0^{1/\lambda}$$

$$= -N_0[e^{-\lambda(1/\lambda)} - e^0] = -N_0[e^{-1} - e^0]$$

$$= +N_0[e^0 - e^{-1}] = N_0[1 - e^{-1}] = \mathbf{0.632\,N_0}$$

Now try the following exercise

Exercise 194 Further problems on mean or average values

1. Determine the mean value of (a) $y = 3\sqrt{x}$ from $x = 0$ to $x = 4$ (b) $y = \sin 2\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{4}$ (c) $y = 4e^t$ from $t = 1$ to $t = 4$
$\left[\text{(a) } 4 \quad \text{(b) } \frac{2}{\pi} \text{ or } 0.637 \quad \text{(c) } 69.17\right]$

2. Calculate the mean value of $y = 2x^2 + 5$ in the range $x = 1$ to $x = 4$ by (a) the mid-ordinate rule, and (b) integration. [19]

3. The speed v of a vehicle is given by: $v=(4t+3)$ m/s, where t is the time in seconds. Determine the average value of the speed from $t=0$ to $t=3$ s. [9 m/s]

4. Find the mean value of the curve $y=6+x-x^2$ which lies above the x-axis by using an approximate method. Check the result using integration. [4.17]

5. The vertical height h km of a missile varies with the horizontal distance d km, and is given by $h=4d-d^2$. Determine the mean height of the missile from $d=0$ to $d=4$ km. [2.67 km]

6. The velocity v of a piston moving with simple harmonic motion at any time t is given by: $v=c\sin\omega t$, where c is a constant. Determine the mean velocity between $t=0$ and $t=\dfrac{\pi}{\omega}$ $\left[\dfrac{2c}{\pi}\right]$

56.2 Root mean square values

The **root mean square value** of a quantity is 'the sqaure root of the mean value of the squared values of the quantity' taken over an interval. With reference to Fig. 56.1, the r.m.s. value of $y=f(x)$ over the range $x=a$ to $x=b$ is given by:

$$\textbf{r.m.s. value} = \sqrt{\frac{1}{b-a}\int_a^b y^2\,dx}$$

One of the principal applications of r.m.s. values is with alternating currents and voltages. The r.m.s. value of an alternating current is defined as that current which will give the same heating effect as the equivalent direct current.

Problem 5. Determine the r.m.s. value of $y=2x^2$ between $x=1$ and $x=4$

R.m.s. value

$$= \sqrt{\frac{1}{4-1}\int_1^4 y^2\,dx} = \sqrt{\frac{1}{3}\int_1^4 (2x^2)^2\,dx}$$

$$= \sqrt{\frac{1}{3}\int_1^4 4x^4\,dx} = \sqrt{\frac{4}{3}\left[\frac{x^5}{5}\right]_1^4}$$

$$= \sqrt{\frac{4}{15}(1024-1)} = \sqrt{272.8} = \mathbf{16.5}$$

Problem 6. A sinusoidal voltage has a maximum value of 100 V. Calculate its r.m.s. value

A sinusoidal voltage v having a maximum value of 100 V may be written as: $v=100\sin\theta$. Over the range $\theta=0$ to $\theta=\pi$,

r.m.s. value

$$= \sqrt{\frac{1}{\pi-0}\int_0^\pi v^2\,d\theta}$$

$$= \sqrt{\frac{1}{\pi}\int_0^\pi (100\sin\theta)^2\,d\theta}$$

$$= \sqrt{\frac{10\,000}{\pi}\int_0^\pi \sin^2\theta\,d\theta}$$

which is not a 'standard' integral. It is shown in Chapter 27 that $\cos 2A = 1-2\sin^2 A$ and this formula is used whenever $\sin^2 A$ needs to be integrated.
Rearranging $\cos 2A = 1-2\sin^2 A$ gives
$\sin^2 A = \frac{1}{2}(1-\cos 2A)$

Hence $\sqrt{\dfrac{10\,000}{\pi}\displaystyle\int_0^\pi \sin^2\theta\,d\theta}$

$$= \sqrt{\frac{10\,000}{\pi}\int_0^\pi \frac{1}{2}(1-\cos 2\theta)\,d\theta}$$

$$= \sqrt{\frac{10\,000}{\pi}\frac{1}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^\pi}$$

$$= \sqrt{\frac{10\,000}{\pi}\frac{1}{2}\left[\left(\pi-\frac{\sin 2\pi}{2}\right)-\left(0-\frac{\sin 0}{2}\right)\right]}$$

$$= \sqrt{\frac{10\,000}{\pi}\frac{1}{2}[\pi]} = \sqrt{\frac{10\,000}{2}}$$

$$= \frac{100}{\sqrt{2}} = \mathbf{70.71\ volts}$$

[Note that for a sine wave,

$$\textbf{r.m.s. value} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\ \textbf{× maximum value}.$$

In this case, r.m.s. value $=\dfrac{1}{\sqrt{2}}\times 100 = 70.71$ V]

Problem 7. In a frequency distribution the average distance from the mean, y, is related to the variable, x, by the equation $y = 2x^2 - 1$. Determine, correct to 3 significant figures, the r.m.s. deviation from the mean for values of x from -1 to $+4$

R.m.s. deviation

$$= \sqrt{\frac{1}{4 - -1}\int_{-1}^{4} y^2\,dx}$$

$$= \sqrt{\frac{1}{5}\int_{-1}^{4} (2x^2 - 1)^2 dx}$$

$$= \sqrt{\frac{1}{5}\int_{-1}^{4} (4x^4 - 4x^2 + 1) dx}$$

$$= \sqrt{\frac{1}{5}\left[\frac{4x^5}{5} - \frac{4x^3}{3} + x\right]_{-1}^{4}}$$

$$= \sqrt{\frac{1}{5}\left[\left(\frac{4}{5}(4)^5 - \frac{4}{3}(4)^3 + 4\right) - \left(\frac{4}{5}(-1)^5 - \frac{4}{3}(-1)^3 + (-1)\right)\right]}$$

$$= \sqrt{\frac{1}{5}[(737.87) - (-0.467)]}$$

$$= \sqrt{\frac{1}{5}[738.34]}$$

$= \sqrt{147.67} = 12.152 = \mathbf{12.2}$,

correct to 3 significant figures.

Now try the following exercise

Exercise 195 Further problems on root mean square values

1. Determine the r.m.s. values of:
   (a) $y = 3x$ from $x = 0$ to $x = 4$
   (b) $y = t^2$ from $t = 1$ to $t = 3$
   (c) $y = 25 \sin\theta$ from $\theta = 0$ to $\theta = 2\pi$

   $\left[\text{(a) } 6.928 \quad \text{(b) } 4.919 \quad \text{(c) } \frac{25}{\sqrt{2}} \text{ or } 17.68\right]$

2. Calculate the r.m.s. values of:
   (a) $y = \sin 2\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{4}$
   (b) $y = 1 + \sin t$ from $t = 0$ to $t = 2\pi$
   (c) $y = 3\cos 2x$ from $x = 0$ to $x = \pi$

   (Note that $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$, from Chapter 27).

   $\left[\text{(a) } \frac{1}{\sqrt{2}} \text{ or } 0.707 \quad \text{(b) } 1.225 \quad \text{(c) } 2.121\right]$

3. The distance, p, of points from the mean value of a frequency distribution are related to the variable, q, by the equation $p = \frac{1}{q} + q$. Determine the standard deviation (i.e. the r.m.s. value), correct to 3 significant figures, for values from $q = 1$ to $q = 3$. [2.58]

4. A current, $i = 30 \sin 100\pi t$ amperes is applied across an electric circuit. Determine its mean and r.m.s. values, each correct to 4 significant figures, over the range $t = 0$ to $t = 10$ ms. [19.10 A, 21.21 A]

5. A sinusoidal voltage has a peak value of 340 V. Calculate its mean and r.m.s. values, correct to 3 significant figures. [216 V, 240 V]

6. Determine the form factor, correct to 3 significant figures, of a sinusoidal voltage of maximum value 100 volts, given that form factor $= \dfrac{\text{r.m.s. value}}{\text{average value}}$ [1.11]

7. A wave is defined by the equation:

   $$v = E_1 \sin\omega t + E_3 \sin 3\omega t$$

   where, E_1, E_3 and ω are constants.

   Determine the r.m.s. value of v over the interval $0 \le t \le \frac{\pi}{\omega}$

   $\left[\sqrt{\frac{E_1^2 + E_3^2}{2}}\right]$

Chapter 57

Volumes of solids of revolution

57.1 Introduction

If the area under the curve $y = f(x)$, (shown in Fig. 57.1(a)), between $x = a$ and $x = b$ is rotated 360° about the x-axis, then a volume known as a **solid of revolution** is produced as shown in Fig. 57.1(b).

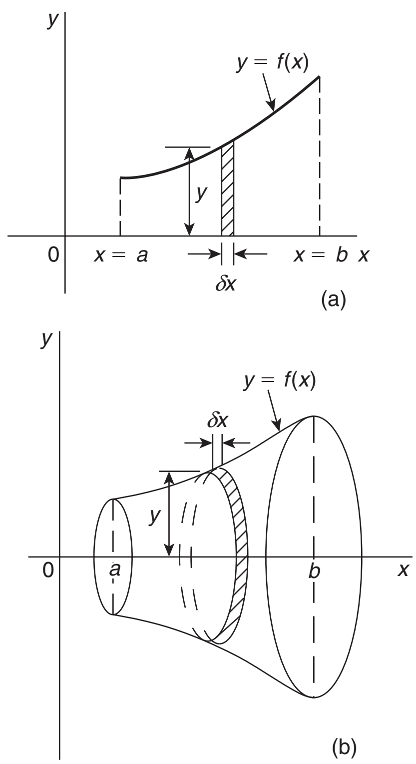

Figure 57.1

The volume of such a solid may be determined precisely using integration.

(i) Let the area shown in Fig. 57.1(a) be divided into a number of strips each of width δx. One such strip is shown shaded.

(ii) When the area is rotated 360° about the x-axis, each strip produces a solid of revolution approximating to a circular disc of radius y and thickness δx. Volume of disc = (circular cross-sectional area) (thickness) = $(\pi y^2)(\delta x)$

(iii) Total volume, V, between ordinates $x = a$ and $x = b$ is given by:

$$\textbf{Volume } V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x = \int_a^b \pi y^2 \, dx$$

If a curve $x = f(y)$ is rotated about the y-axis 360° between the limits $y = c$ and $y = d$, as shown in Fig. 57.2, then the volume generated is given by:

$$\textbf{Volume } V = \lim_{\delta y \to 0} \sum_{y=c}^{y=d} \pi x^2 \delta y = \int_c^d \pi x^2 \, dy$$

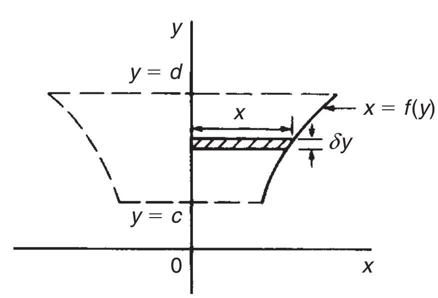

Figure 57.2

57.2 Worked problems on volumes of solids of revolution

Problem 1. Determine the volume of the solid of revolution formed when the curve $y=2$ is rotated 360° about the x-axis between the limits $x=0$ to $x=3$

When $y=2$ is rotated 360° about the x-axis between $x=0$ and $x=3$ (see Fig. 57.3):

volume generated

$$=\int_0^3 \pi y^2\,dx=\int_0^3 \pi(2)^2\,dx$$

$$=\int_0^3 4\pi\,dx=4\pi[x]_0^3=\mathbf{12\pi\ cubic\ units}$$

[Check: The volume generated is a cylinder of radius 2 and height 3.
Volume of cylinder $=\pi r^2h=\pi(2)^2(3)=$ **12π cubic units**.]

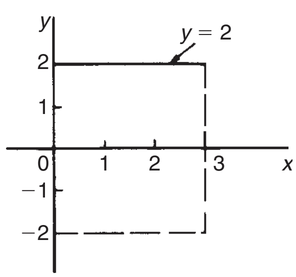

Figure 57.3

Problem 2. Find the volume of the solid of revolution when the cure $y=2x$ is rotated one revolution about the x-axis between the limits $x=0$ and $x=5$

When $y=2x$ is revolved one revolution about the x-axis between $x=0$ and $x=5$ (see Fig. 57.4) then:

volume generated

$$=\int_0^5 \pi y^2 dx=\int_0^5 \pi(2x)^2 dx$$

$$=\int_0^5 4\pi x^2 dx=4\pi\left[\frac{x^3}{3}\right]_0^5$$

$$=\frac{500\pi}{3}=\mathbf{166\frac{2}{3}\pi\ cubic\ units}$$

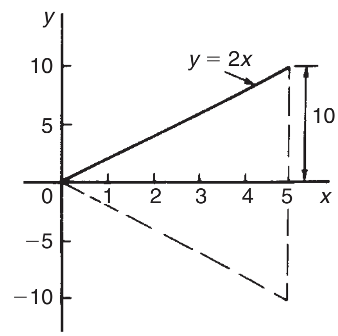

Figure 57.4

[Check: The volume generated is a cone of radius 10 and height 5. Volume of cone

$$=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi(10)^2 5=\frac{500\pi}{3}$$

$$=\mathbf{166\frac{2}{3}\pi\ cubic\ units}.]$$

Problem 3. The curve $y=x^2+4$ is rotated one revolution about the x-axis between the limits $x=1$ and $x=4$. Determine the volume of the solid of revolution produced

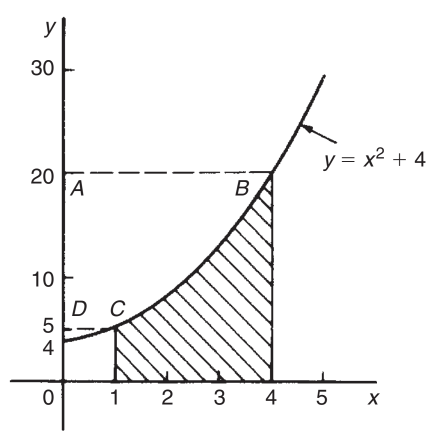

Figure 57.5

Revolving the shaded area shown in Fig. 57.5 about the x-axis 360° produces a solid of revolution given by:

$$\text{Volume}=\int_1^4 \pi y^2\,dx=\int_1^4 \pi(x^2+4)^2 dx$$

$$=\int_1^4 \pi(x^4+8x^2+16)\,dx$$

$$= \pi\left[\frac{x^5}{5} + \frac{8x^3}{3} + 16x\right]_1^4$$
$$= \pi[(204.8 + 170.67 + 64) - (0.2 + 2.67 + 16)]$$
$$= \mathbf{420.6\pi \text{ cubic units}}$$

Problem 4. If the curve in Problem 3 is revolved about the y-axis between the same limits, determine the volume of the solid of revolution produced

The volume produced when the curve $y = x^2 + 4$ is rotated about the y-axis between $y = 5$ (when $x = 1$) and $y = 20$ (when $x = 4$), i.e. rotating area ABCD of Fig. 57.5 about the y-axis is given by:

$$\text{volume} = \int_5^{20} \pi x^2\, dy$$

Since $y = x^2 + 4$, then $x^2 = y - 4$

$$\text{Hence volume} = \int_5^{20} \pi(y-4)dy = \pi\left[\frac{y^2}{2} - 4y\right]_5^{20}$$
$$= \pi[(120) - (-7.5)]$$
$$= \mathbf{127.5\pi \text{ cubic units}}$$

Now try the following exercise

Exercise 196 Further problems on volumes of solids of revolution

(Answers are in cubic units and in terms of π).

In Problems 1 to 5, determine the volume of the solid of revolution formed by revolving the areas enclosed by the given curve, the x-axis and the given ordinates through one revolution about the x-axis.

1. $y = 5x; \quad x = 1, x = 4$ [525π]
2. $y = x^2; \quad x = -2, x = 3$ [55π]
3. $y = 2x^2 + 3; \quad x = 0, x = 2$ [75.6π]
4. $\frac{y^2}{4} = x; \quad x = 1, x = 5$ [48π]
5. $xy = 3; \quad x = 2, x = 3$ [1.5π]

In Problems 6 to 8, determine the volume of the solid of revolution formed by revolving the areas enclosed by the given curves, the y-axis and the given ordinates through one revolution about the y-axis.

6. $y = x^2; \quad y = 1, y = 3$ [4π]
7. $y = 3x^2 - 1; \quad y = 2, y = 4$ [2.67π]
8. $y = \frac{2}{x}; \quad y = 1, y = 3$ [2.67π]
9. The curve $y = 2x^2 + 3$ is rotated about (a) the x-axis between the limits $x = 0$ and $x = 3$, and (b) the y-axis, between the same limits. Determine the volume generated in each case. [(a) 329.4π (b) 81π]

57.3 Further worked problems on volumes of solids of revolution

Problem 5. The area enclosed by the curve $y = 3e^{\frac{x}{3}}$, the x-axis and ordinates $x = -1$ and $x = 3$ is rotated 360° about the x-axis. Determine the volume generated

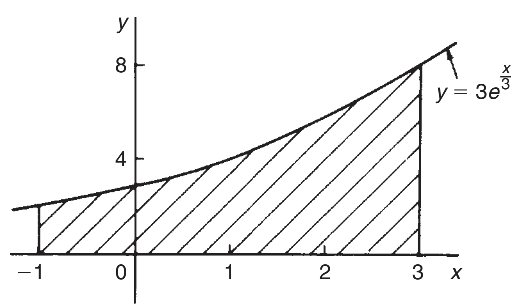

Figure 57.6

A sketch of $y = 3e^{\frac{x}{3}}$ is shown in Fig. 57.6. When the shaded area is rotated 360° about the x-axis then:

$$\text{volume generated} = \int_{-1}^{3} \pi y^2\, dx$$
$$= \int_{-1}^{3} \pi\left(3e^{\frac{x}{3}}\right)^2 dx$$
$$= 9\pi \int_{-1}^{3} e^{\frac{2x}{3}}\, dx$$

$$= 9\pi\left[\frac{e^{\frac{2x}{3}}}{\frac{2}{3}}\right]_{-1}^{3}$$

$$= \frac{27\pi}{2}\left(e^2 - e^{-\frac{2}{3}}\right)$$

$$= \mathbf{92.82\pi} \textit{ cubic units}$$

Problem 6. Determine the volume generated when the area above the x-axis bounded by the curve $x^2 + y^2 = 9$ and the ordinates $x = 3$ and $x = -3$ is rotated one revolution about the x-axis

Figure 57.7 shows the part of the curve $x^2 + y^2 = 9$ lying above the x-axis, Since, in general, $x^2 + y^2 = r^2$ represents a circle, centre 0 and radius r, then $x^2 + y^2 = 9$ represents a circle, centre 0 and radius 3. When the semi-circular area of Fig. 57.7 is rotated one revolution about the x-axis then:

$$\text{volume generated} = \int_{-3}^{3} \pi y^2 dx$$

$$= \int_{-3}^{3} \pi(9 - x^2)\, dx$$

$$= \pi\left[9x - \frac{x^3}{3}\right]_{-3}^{3}$$

$$= \pi[(18) - (-18)]$$

$$= \mathbf{36\pi \text{ cubic units}}$$

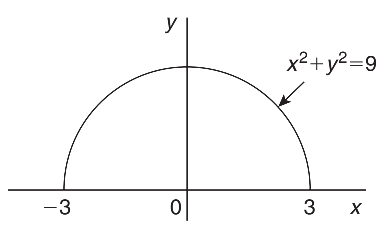

Figure 57.7

(Check: The volume generated is a sphere of radius 3. Volume of sphere $= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3)^3 =$ **36π cubic units**.)

Problem 7. Calculate the volume of a frustum of a sphere of radius 4 cm that lies between two parallel planes at 1 cm and 3 cm from the centre and on the same side of it

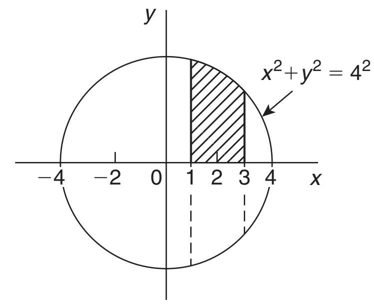

Figure 57.8

The volume of a frustum of a sphere may be determined by integration by rotating the curve $x^2 + y^2 = 4^2$ (i.e. a circle, centre 0, radius 4) one revolution about the x-axis, between the limits $x = 1$ and $x = 3$ (i.e. rotating the shaded area of Fig. 57.8).

$$\text{Volume of frustum} = \int_{1}^{3} \pi y^2\, dx$$

$$= \int_{1}^{3} \pi(4^2 - x^2)\, dx$$

$$= \pi\left[16x - \frac{x^3}{3}\right]_{1}^{3}$$

$$= \pi\left[(39) - \left(15\frac{2}{3}\right)\right]$$

$$= \mathbf{23\frac{1}{3}\pi \text{ cubic units}}$$

Problem 8. The area enclosed between the two parabolas $y = x^2$ and $y^2 = 8x$ of Problem 12, Chapter 55, page 485, is rotated 360° about the x-axis. Determine the volume of the solid produced

The area enclosed by the two curves is shown in Fig. 55.13, page 485. The volume produced by revolving the shaded area about the x-axis is given by: [(volume produced by revolving $y^2 = 8x$) − (volume produced by revolving $y = x^2$)]

$$\text{i.e.} \quad \textbf{volume} = \int_0^2 \pi(8x)\, dx - \int_0^2 \pi(x^4)\, dx$$

$$= \pi\int_0^2 (8x - x^4)\, dx = \pi\left[\frac{8x^2}{2} - \frac{x^5}{5}\right]_0^2$$

$$= \pi\left[\left(16 - \frac{32}{5}\right) - (0)\right]$$

$$= \mathbf{9.6\pi \text{ cubic units}}$$

Now try the following exercise

Exercise 197 Further problems on volumes of solids of revolution

(Answers to volumes are in cubic units and in terms of π.)

In Problems 1 and 2, determine the volume of the solid of revolution formed by revolving the areas enclosed by the given curve, the x-axis and the given ordinates through one revolution about the x-axis.

1. $y = 4e^x$; $x = 0$; $x = 2$ [428.8π]

2. $y = \sec x$; $x = 0$, $x = \dfrac{\pi}{4}$ [π]

In Problems 3 and 4, determine the volume of the solid of revolution formed by revolving the areas enclosed by the given curves, the y-axis and the given ordinates through one revolution about the y-axis.

3. $x^2 + y^2 = 16$; $y = 0$, $y = 4$ [42.67π]

4. $x\sqrt{y} = 2$; $y = 2$, $y = 3$ [1.622π]

5. Determine the volume of a plug formed by the frustum of a sphere of radius 6 cm which lies between two parallel planes at 2 cm and 4 cm from the centre and on the same side of it. (The equation of a circle, centre 0, radius r is $x^2 + y^2 = r^2$). [53.33π]

6. The area enclosed between the two curves $x^2 = 3y$ and $y^2 = 3x$ is rotated about the x-axis. Determine the volume of the solid formed. [8.1π]

7. The portion of the curve $y = x^2 + \dfrac{1}{x}$ lying between $x = 1$ and $x = 3$ is revolved 360° about the x-axis. Determine the volume of the solid formed. [57.07π]

8. Calculate the volume of the frustum of a sphere of radius 5 cm that lies between two parallel planes at 3 cm and 2 cm from the centre and on opposite sides of it. [113.33π]

9. Sketch the curves $y = x^2 + 2$ and $y - 12 = 3x$ from $x = -3$ to $x = 6$. Determine (a) the co-ordinates of the points of intersection of the two curves, and (b) the area enclosed by the two curves. (c) If the enclosed area is rotated 360° about the x-axis, calculate the volume of the solid produced

   [(a) $(-2, 6)$ and $(5, 27)$
   (b) 57.17 square units
   (c) 1326π cubic units]

Chapter 58

Centroids of simple shapes

58.1 Centroids

A **lamina** is a thin flat sheet having uniform thickness. The **centre of gravity** of a lamina is the point where it balances perfectly, i.e. the lamina's **centre of mass**. When dealing with an area (i.e. a lamina of negligible thickness and mass) the term **centre of area** or **centroid** is used for the point where the centre of gravity of a lamina of that shape would lie.

58.2 The first moment of area

The **first moment of area** is defined as the product of the area and the perpendicular distance of its centroid from a given axis in the plane of the area. In Fig. 58.1, the first moment of area A about axis XX is given by (Ay) cubic units.

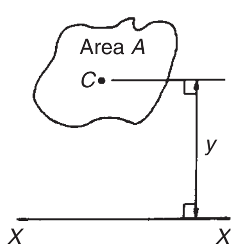

Figure 58.1

58.3 Centroid of area between a curve and the x-axis

(i) Figure 58.2 shows an area $PQRS$ bounded by the curve $y = f(x)$, the x-axis and ordinates $x = a$ and $x = b$. Let this area be divided into a large number of strips, each of width δx. A typical strip is shown shaded drawn at point (x, y) on $f(x)$. The area of the strip is approximately rectangular and is given by $y\delta x$. The centroid, C, has coordinates $\left(x, \dfrac{y}{2}\right)$.

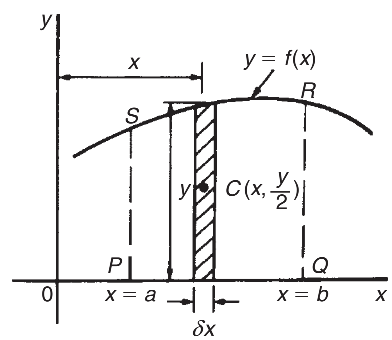

Figure 58.2

(ii) First moment of area of shaded strip about axis $Oy = (y\delta x)(x) = xy\delta x$.
Total first moment of area $PQRS$ about axis $Oy = \underset{\delta x \to 0}{\text{limit}} \sum_{x=a}^{x=b} xy\delta x = \int_a^b xy\,dx$

(iii) First moment of area of shaded strip about axis $Ox = (y\delta x)\left(\dfrac{y}{2}\right) = \dfrac{1}{2}y^2 x.$
Total first moment of area $PQRS$ about axis $Ox = \underset{\delta x \to 0}{\text{limit}} \sum_{x=a}^{x=b} \dfrac{1}{2}y^2\delta x = \dfrac{1}{2}\displaystyle\int_a^b y^2\,dx$

(iv) Area of $PQRS$, $A = \int_a^b y\,dx$ (from Chapter 55)

(v) Let $\bar{x}$ and $\bar{y}$ be the distances of the centroid of area A about Oy and Ox respectively then: $(\bar{x})(A) =$ total first moment of area A about axis $Oy = \int_a^b xy\,dx$

from which, $$\bar{x} = \frac{\displaystyle\int_a^b xy\,dx}{\displaystyle\int_a^b y\,dx}$$

and $(\bar{y})(A)$ = total moment of area A about axis $Ox = \frac{1}{2}\int_a^b y^2\,dx$

from which, $$\bar{y} = \frac{\frac{1}{2}\int_a^b y^2\,dx}{\int_a^b y\,dx}$$

58.4 Centroid of area between a curve and the y-axis

If $\bar{x}$ and $\bar{y}$ are the distances of the centroid of area EFGH in Fig. 58.3 from Oy and Ox respectively, then, by similar reasoning as above:

$$(\bar{x})(\text{total area}) = \lim_{\delta y \to 0} \sum_{y=c}^{y=d} x\delta y\left(\frac{x}{2}\right) = \frac{1}{2}\int_c^d x^2 dy$$

from which, $$\bar{x} = \frac{\frac{1}{2}\int_c^d x^2 dy}{\int_c^d x\,dy}$$

and $$(\bar{y})(\text{total area}) = \lim_{\delta y \to 0} \sum_{y=c}^{y=d} (x\delta y)y = \int_c^d xy\,dy$$

from which, $$\bar{y} = \frac{\int_c^d xy\,dy}{\int_c^d x\,dy}$$

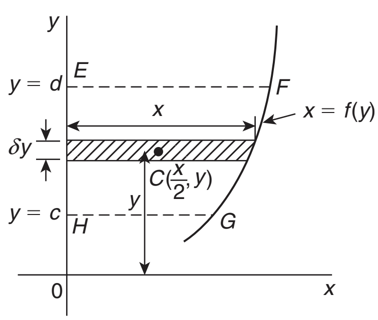

Figure 58.3

58.5 Worked problems on centroids of simple shapes

Problem 1. Show, by integration, that the centroid of a rectangle lies at the intersection of the diagonals

Let a rectangle be formed by the line $y = b$, the x-axis and ordinates $x = 0$ and $x = l$ as shown in Fig. 58.4. Let the coordinates of the centroid C of this area be $(\bar{x}, \bar{y})$.

By integration, $$\bar{x} = \frac{\int_0^l xy\,dx}{\int_0^l y\,dx} = \frac{\int_0^l (x)(b)\,dx}{\int_0^l b\,dx}$$

$$= \frac{\left[b\frac{x^2}{2}\right]_0^l}{[bx]_0^l} = \frac{\frac{bl^2}{2}}{bl} = \frac{1}{2}$$

and $$\bar{y} = \frac{\frac{1}{2}\int_0^l y^2\,dx}{\int_0^l y\,dx} = \frac{\frac{1}{2}\int_0^l b^2\,dx}{bl}$$

$$= \frac{\frac{1}{2}[b^2x]_0^l}{bl} = \frac{\frac{b^2l}{2}}{bl} = \frac{b}{2}$$

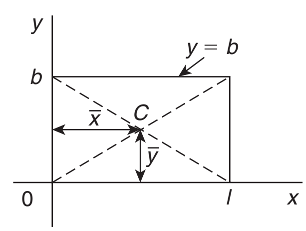

Figure 58.4

i.e. **the centroid lies at $\left(\frac{l}{2}, \frac{b}{2}\right)$ which is at the intersection of the diagonals.**

Problem 2. Find the position of the centroid of the area bounded by the curve $y = 3x^2$, the x-axis and the ordinates $x = 0$ and $x = 2$

If, $(\bar{x}, \bar{y})$ are the co-ordinates of the centroid of the given area then:

$$\bar{x} = \frac{\int_0^2 xy\,dx}{\int_0^2 y\,dx} = \frac{\int_0^2 x(3x^2)dx}{\int_0^2 3x^2 dx}$$

$$= \frac{\int_0^2 3x^3 dx}{\int_0^2 3x^2 dx} = \frac{\left[\frac{3x^4}{4}\right]_0^2}{[x^3]_0^2} = \frac{12}{8} = \mathbf{1.5}$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^2 y^2 dx}{\int_0^2 y dx} = \frac{\frac{1}{2}\int_0^2 (3x^2)^2 dx}{8}$$

$$= \frac{\frac{1}{2}\int_0^2 9x^4 dx}{8} = \frac{\frac{9}{2}\left[\frac{x^5}{5}\right]_0^2}{8} = \frac{\frac{9}{2}\left(\frac{32}{5}\right)}{8}$$

$$= \frac{18}{5} = \mathbf{3.6}$$

Hence the centroid lies at (1.5, 3.6)

Problem 3. Determine by integration the position of the centroid of the area enclosed by the line $y = 4x$, the x-axis and ordinates $x = 0$ and $x = 3$

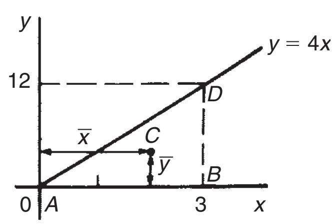

Figure 58.5

Let the coordinates of the area be $(\bar{x}, \bar{y})$ as shown in Fig. 58.5.

Then $$\bar{x} = \frac{\int_0^3 xy\, dx}{\int_0^3 y\, dx} = \frac{\int_0^3 (x)(4x) dx}{\int_0^3 4x\, dx}$$

$$= \frac{\int_0^3 4x^2 dx}{\int_0^3 4x\, dx} = \frac{\left[\frac{4x^3}{3}\right]_0^3}{[2x^2]_0^3} = \frac{36}{18} = 2$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^3 y^2 dx}{\int_0^3 y\, dx} = \frac{\frac{1}{2}\int_0^3 (4x)^2 dx}{18}$$

$$= \frac{\frac{1}{2}\int_0^3 16x^2 dx}{18} = \frac{\frac{1}{2}\left[\frac{16x^3}{3}\right]_0^3}{18} = \frac{72}{18} = 4$$

Hence the centroid lies at (2, 4).

In Fig. 58.5, *ABD* is a right-angled triangle. The centroid lies 4 units from *AB* and 1 unit from *BD* showing that the centroid of a triangle lies at one-third of the perpendicular height above any side as base.

Now try the following exercise

Exercise 198 Further problems on centroids of simple shapes

In Problems 1 to 5, find the position of the centroids of the areas bounded by the given curves, the x-axis and the given ordinates.

1. $y = 2x;\ x = 0, x = 3$ [(2, 2)]
2. $y = 3x + 2;\ x = 0, x = 4$ [(2.50, 4.75)]
3. $y = 5x^2;\ x = 1, x = 4$ [(3.036, 24.36)]
4. $y = 2x^3;\ x = 0, x = 2$ [(1.60, 4.57)]
5. $y = x(3x + 1);\ x = -1, x = 0$ [(−0.833, 0.633)]

58.6 Further worked problems on centroids of simple shapes

Problem 4. Determien the co-ordinates of the centroid of the area lying between the curve $y = 5x - x^2$ and the x-axis

$y = 5x - x^2 = x(5 - x)$. When $y = 0$, $x = 0$ or $x = 5$, Hence the curve cuts the x-axis at 0 and 5 as shown in Fig. 58.6. Let the co-ordinates of the centroid be $(\bar{x}, \bar{y})$ then, by integration,

$$\bar{x} = \frac{\int_0^5 xy\, dx}{\int_0^5 y\, dx} = \frac{\int_0^5 x(5x - x^2) dx}{\int_0^5 (5x - x^2) dx}$$

$$= \frac{\int_0^5 (5x^2 - x^3) dx}{\int_0^5 (5x - x^2) dx} = \frac{\left[\frac{5x^3}{3} - \frac{x^4}{4}\right]_0^5}{\left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5}$$

$$= \frac{\frac{625}{3} - \frac{625}{4}}{\frac{125}{2} - \frac{125}{3}} = \frac{\frac{625}{12}}{\frac{125}{6}}$$

$$= \left(\frac{625}{12}\right)\left(\frac{6}{125}\right) = \frac{5}{2} = \mathbf{2.5}$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^5 y^2 dx}{\int_0^5 y\,dx} = \frac{\frac{1}{2}\int_0^5 (5x - x^2)^2 dx}{\int_0^5 (5x - x^2)\,dx}$$

$$= \frac{\frac{1}{2}\int_0^5 (25x^2 - 10x^3 + x^4)\,dx}{\frac{125}{6}}$$

$$= \frac{\frac{1}{2}\left[\frac{25x^3}{3} - \frac{10x^4}{4} + \frac{x^5}{5}\right]_0^5}{\frac{125}{6}}$$

$$= \frac{\frac{1}{2}\left(\frac{25(125)}{3} - \frac{6250}{4} + 625\right)}{\frac{125}{6}} = \mathbf{2.5}$$

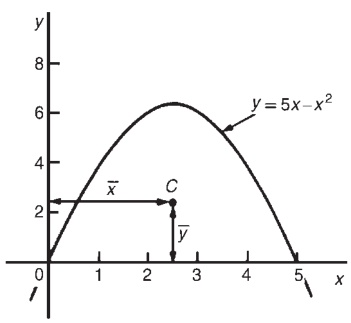

Figure 58.6

Hence the centroid of the area lies at (2.5, 2.5)
(Note from Fig. 58.6 that the curve is symmetrical about $x = 2.5$ and thus $\bar{x}$ could have been determined 'on sight').

Problem 5. Locate the centroid of the area enclosed by the curve $y = 2x^2$, the y-axis and ordinates $y = 1$ and $y = 4$, correct to 3 decimal places

From Section 58.4,

$$\bar{x} = \frac{\frac{1}{2}\int_1^4 x^2 dy}{\int_1^4 x\,dy} = \frac{\frac{1}{2}\int_1^4 \frac{y}{2} dy}{\int_1^4 \sqrt{\frac{y}{2}} dy}$$

$$= \frac{\frac{1}{2}\left[\frac{y^2}{4}\right]_1^4}{\left[\frac{2y^{3/2}}{3\sqrt{2}}\right]_1^4} = \frac{\frac{15}{8}}{\frac{14}{3\sqrt{2}}} = \mathbf{0.568}$$

and $$\bar{y} = \frac{\int_1^4 xy\,dy}{\int_1^4 x\,dy} = \frac{\int_1^4 \sqrt{\frac{y}{2}}(y)\,dy}{\frac{14}{3\sqrt{2}}}$$

$$= \frac{\int_1^4 \frac{y^{3/2}}{\sqrt{2}} dy}{\frac{14}{3\sqrt{2}}} = \frac{\frac{1}{\sqrt{2}}\left[\frac{y^{5/2}}{\frac{5}{2}}\right]_1^4}{\frac{14}{3\sqrt{2}}}$$

$$= \frac{\frac{2}{5\sqrt{2}}(31)}{\frac{14}{3\sqrt{2}}} = \mathbf{2.657}$$

Hence the position of the centroid is at (0.568, 2.657)

Problem 6. Locate the position of the centroid enclosed by the curves $y = x^2$ and $y^2 = 8x$

Figure 58.7 shows the two curves intersection at (0, 0) and (2, 4). These are the same curves as used in Problem 12, Chapter 55 where the shaded area was

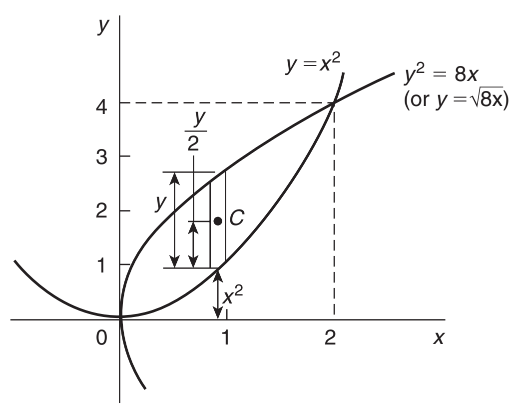

Figure 58.7

calculated as $2\frac{2}{3}$ square units. Let the co-ordinates of centroid C be $\bar{x}$ and $\bar{y}$.

By integration, $\bar{x} = \dfrac{\int_0^2 xy\,dx}{\int_0^2 y\,dx}$

The value of y is given by the height of the typical strip shown in Fig. 58.7, i.e. $y = \sqrt{8x} - x^2$. Hence,

$$\bar{x} = \frac{\int_0^2 x(\sqrt{8x} - x^2)dx}{2\frac{2}{3}} = \frac{\int_0^2 (\sqrt{8}x^{3/2} - x^3)}{2\frac{2}{3}}$$

$$= \frac{\left[\sqrt{8}\frac{x^{5/2}}{\frac{5}{2}} - \frac{x^4}{4}\right]_0^2}{2\frac{2}{3}} = \left(\frac{\sqrt{8}\frac{\sqrt{2^5}}{\frac{5}{2}} - 4}{2\frac{2}{3}}\right)$$

$$= \frac{2\frac{2}{5}}{2\frac{2}{3}} = \mathbf{0.9}$$

Care needs to be taken when finding $\bar{y}$ in such examples as this. From Fig. 58.7, $y = \sqrt{8x} - x^2$ and $\frac{y}{2} = \frac{1}{2}(\sqrt{8x} - x^2)$. The perpendicular distance from centroid C of the strip to Ox is $\frac{1}{2}(\sqrt{8x} - x^2) + x^2$.

Taking moments about Ox gives:

(total area) $(\bar{y}) = \sum_{x=0}^{x=2}$(area of strip) (perpendicular distance of centroid of strip to Ox)

Hence (area) $(\bar{y})$

$$= \int \left[\sqrt{8x} - x^2\right]\left[\frac{1}{2}(\sqrt{8x} - x^2) + x^2\right]dx$$

i.e. $\left(2\frac{2}{3}\right)(\bar{y}) = \int_0^2 \left[\sqrt{8x} - x^2\right]\left(\frac{\sqrt{8x}}{2} + \frac{x^2}{2}\right)dx$

$$= \int_0^2 \left(\frac{8x}{2} - \frac{x^4}{2}\right)dx = \left[\frac{8x^2}{4} - \frac{x^5}{10}\right]_0^2$$

$$= \left(8 - 3\frac{1}{5}\right) - (0) = 4\frac{4}{5}$$

Hence $\quad \bar{y} = \dfrac{4\frac{4}{5}}{2\frac{2}{3}} = \mathbf{1.8}$

Thus the position of the centroid of the enclosed area in Fig. 58.7 is at (0.9, 1.8)

Now try the following exercise

Exercise 199 Further problems on centroids of simple shapes

1. Determine the position of the centroid of a sheet of metal formed by the curve $y = 4x - x^2$ which lies above the x-axis. [(2, 1.6)]

2. Find the coordinates of the centroid of the area that lies between curve $\frac{y}{x} = x - 2$ and the x-axis. [(1, −0.4)]

3. Determine the coordinates of the centroid of the area formed between the curve $y = 9 - x^2$ and the x-axis. [(0, 3.6)]

4. Determine the centroid of the area lying between $y = 4x^2$, the y-axis and the ordinates $y = 0$ and $y = 4$. [(0.375, 2.40]

5. Find the position of the centroid of the area enclosed by the curve $y = \sqrt{5x}$, the x-axis and the ordinate $x = 5$. [(3.0, 1.875)]

6. Sketch the curve $y^2 = 9x$ between the limits $x = 0$ and $x = 4$. Determine the position of the centroid of this area. [(2.4, 0)]

7. Calculate the points of intersection of the curves $x^2 = 4y$ and $\frac{y^2}{4} = x$, and determine the position of the centroid of the area enclosed by them. [(0, 0) and (4, 4), (1.80, 1.80)]

8. Determine the position of the centroid of the sector of a circle of radius 3 cm whose angle subtended at the centre is 40°. [On the centre line, 1.96 cm from the centre]

9. Sketch the curves $y = 2x^2 + 5$ and $y - 8 = x(x + 2)$ on the same axes and determine their points of intersection. Calculate the coordinates of the centroid of the area enclosed by the two curves. [(−1, 7) and (3, 23), (1, 10.20)]

58.7 Theorem of Pappus

A theorem of Pappus states:
'If a plane area is rotated about an axis in its own plane but not intersecting it, the volume of the solid formed is given by the product of the area and the distance moved by the centroid of the area'.

With reference to Fig. 58.8, when the curve $y=f(x)$ is rotated one revolution about the x-axis between the limits $x=a$ and $x=b$, the volume V generated is given by:
volume $V=(A)(2\pi\bar{y})$, from which,

$$\bar{y}=\frac{V}{2\pi A}$$

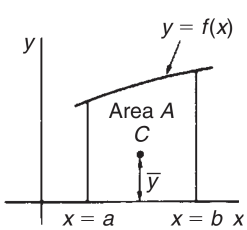

Figure 58.8

Problem 7. Determine the position of the centroid of a semicircle of radius r by using the theorem of Pappus. Check the answer by using integration (given that the equation of a circle, centre 0, radius r is $x^2+y^2=r^2$)

A semicircle is shown in Fig. 58.9 with its diameter lying on the x-axis and its centre at the origin. Area of semicircle $=\frac{\pi r^2}{2}$. When the area is rotated about the x-axis one revolution a sphere is generated of volume $\frac{4}{3}\pi r^3$.

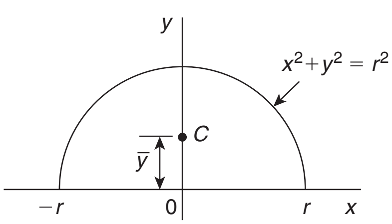

Figure 58.9

Let centroid C be at a distance $\bar{y}$ from the origin as shown in Fig. 58.9. From the theorem of Pappus, volume generated = area × distance moved through by centroid i.e.

$$\frac{4}{3}\pi r^3=\left(\frac{\pi r^2}{2}\right)(2\pi\bar{y})$$

Hence $$\bar{y}=\frac{\frac{4}{3}\pi r^3}{\pi^2 r^2}=\frac{4r}{3\pi}$$

By integration,

$$\bar{y}=\frac{\frac{1}{2}\int_{-r}^{r}y^2dx}{\text{area}}$$

$$=\frac{\frac{1}{2}\int_{-r}^{r}(r^2-x^2)dx}{\frac{\pi r^2}{2}}=\frac{\frac{1}{2}\left[r^2x-\frac{x^3}{3}\right]_{-r}^{r}}{\frac{\pi r^2}{2}}$$

$$=\frac{\frac{1}{2}\left[\left(r^3-\frac{r^3}{3}\right)-\left(-r^3+\frac{r^3}{3}\right)\right]}{\frac{\pi r^2}{2}}=\frac{4r}{3\pi}$$

Hence the centroid of a semicircle lies on the axis of symmetry, distance $\frac{4r}{3\pi}$ (or $0.424r$) from its diameter.

Problem 8. Calculate the area bounded by the curve $y=2x^2$, the x-axis and ordinates $x=0$ and $x=3$. (b) If this area is revolved (i) about the x-axis and (ii) about the y-axis, find the volumes of the solids produced. (c) Locate the position of the centroid using (i) integration, and (ii) the theorem of Pappus

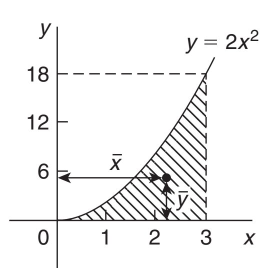

Figure 58.10

(a) The required area is shown shaded in Fig. 58.10.

$$\text{Area} = \int_0^3 y\,dx = \int_0^3 2x^2\,dx = \left[\frac{2x^3}{3}\right]_0^3$$

$$= \mathbf{18 \text{ square units}}$$

(b) (i) When the shaded area of Fig. 58.10 is revolved 360° about the x-axis, the volume generated

$$= \int_0^3 \pi y^2 dx = \int_0^3 \pi(2x^2)^2 dx$$

$$= \int_0^3 4\pi x^4 dx = 4\pi\left[\frac{x^5}{5}\right]_0^3 = 4\pi\left(\frac{243}{5}\right)$$

$$= \mathbf{194.4\pi \text{ cubic units}}$$

(ii) When the shaded area of Fig. 58.10 is revolved 360° about the y-axis, the volume generated = (volume generated by $x = 3$) − (volume generated by $y = 2x^2$)

$$= \int_0^{18} \pi(3)^2 dy - \int_0^{18} \pi\left(\frac{y}{2}\right) dy$$

$$= \pi\int_0^{18}\left(9 - \frac{y}{2}\right) dy = \pi\left[9y - \frac{y^2}{4}\right]_0^{18}$$

$$= \mathbf{81\pi \text{ cubic units}}$$

(c) If the co-ordinates of the centroid of the shaded area in Fig. 58.10 are $(\bar{x}, \bar{y})$ then:

(i) by integration,

$$\bar{x} = \frac{\int_0^3 xy\,dx}{\int_0^3 y\,dx} = \frac{\int_0^3 x(2x^2)\,dx}{18}$$

$$= \frac{\int_0^3 2x^3 dx}{18} = \frac{\left[\frac{2x^4}{4}\right]_0^3}{18} = \frac{81}{36} = \mathbf{2.25}$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^3 y^2 dx}{\int_0^3 y\,dx} = \frac{\frac{1}{2}\int_0^3 (2x^2)^2 dx}{18}$$

$$= \frac{\frac{1}{2}\int_0^3 4x^4 dx}{18} = \frac{\frac{1}{2}\left[\frac{4x^5}{5}\right]_0^3}{18} = \mathbf{5.4}$$

(ii) using the theorem of Pappus:
Volume generated when shaded area is revolved about Oy = (area)($2\pi\bar{x}$)

i.e. $81\pi = (18)(2\pi\bar{x})$,

from which, $\bar{x} = \dfrac{81\pi}{36\pi} = \mathbf{2.25}$

Volume generated when shaded area is revolved about Ox = (area)($2\pi\bar{y}$)

i.e. $194.4\pi = (18)(2\pi\bar{y})$,

from which, $\bar{y} = \dfrac{194.4\pi}{36\pi} = \mathbf{5.4}$

Hence the centroid of the shaded area in Fig. 58.10 is at (2.25, 5.4)

Problem 9. A cylindrical pillar of diameter 400 mm has a groove cut round its circumference. The section of the groove is a semicircle of diameter 50 mm. Determine the volume of material removed, in cubic centimetres, correct to 4 significant figures

A part of the pillar showing the groove is shown in Fig. 58.11.
The distance of the centroid of the semicircle from its base is $\dfrac{4r}{3\pi}$ (see Problem 7) $= \dfrac{4(25)}{3\pi} = \dfrac{100}{3\pi}$ mm.
The distance of the centroid from the centre of the pillar $= \left(200 - \dfrac{100}{3\pi}\right)$ mm.

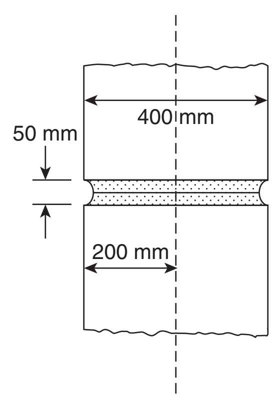

Figure 58.11

The distance moved by the centroid in one revolution

$$= 2\pi\left(200 - \frac{100}{3\pi}\right) = \left(400\pi - \frac{200}{3}\right) \text{mm.}$$

From the theorem of Pappus,
volume = area × distance moved by centroid

$$= \left(\frac{1}{2}\pi 25^2\right)\left(400\pi - \frac{200}{3}\right) = 1168250\,\text{mm}^3$$

Hence the volume of material removed is 1168 cm^3 correct to 4 significant figures.

Problem 10. A metal disc has a radius of 5.0 cm and is of thickness 2.0 cm. A semicircular groove of diameter 2.0 cm is machined centrally around the rim to form a pulley. Determine, using Pappus' theorem, the volume and mass of metal removed and the volume and mass of the pulley if the density of the metal is 8000 kg m^{-3}

A side view of the rim of the disc is shown in Fig. 58.12.

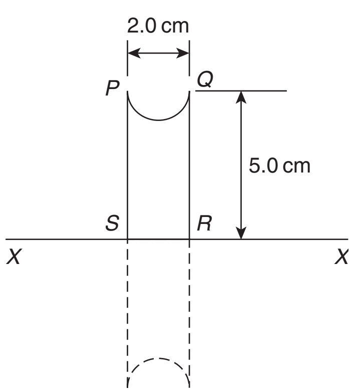

Figure 58.12

When area *PQRS* is rotated about axis *XX* the volume generated is that of the pulley. The centroid of the semicircular area removed is at a distance of $\frac{4r}{3\pi}$ from its diameter (see Problem 7), i.e. $\frac{4(1.0)}{3\pi}$, i.e. 0.424 cm from *PQ*. Thus the distance of the centroid from *XX* is (5.0 − 0.424), i.e. 4.576 cm. The distance moved through in one revolution by the centroid is $2\pi(4.576)$ cm.

Area of semicircle

$$= \frac{\pi r^2}{2} = \frac{\pi(1.0)^2}{2} = \frac{\pi}{2}\,\text{cm}^2$$

By the theorem of Pappus, volume generated
= area × distance moved by centroid

$$= \left(\frac{\pi}{2}\right)(2\pi)(4.576)$$

i.e. **volume of metal removed = 45.16 cm^3**
Mass of metal removed = density × volume

$$= 8000\,\text{kg m}^{-3} \times \frac{45.16}{10^6}\,\text{m}^3$$

$$= \mathbf{0.3613\,kg} \text{ or } \mathbf{361.3\,g}$$

Volume of pulley = volume of cylindrical disc − volume of metal removed

$$= \pi(5.0)^2(2.0) - 45.16 = \mathbf{111.9\,cm^3}$$

Mass of pulley = density × volume

$$= 8000\,\text{kg m}^{-3} \times \frac{111.9}{10^6}\,\text{m}^3$$

$$= \mathbf{0.8952\,kg} \text{ or } \mathbf{895.2\,g}$$

Now try the following exercise

Exercise 200 Further problems on the theorem of Pappus

1. A right angled isosceles triangle having a hypotenuse of 8 cm is revolved one revolution about one of its equal sides as axis. Determine the volume of the solid generated using Pappus' theorem. [189.6 cm^3]

2. A rectangle measuring 10.0 cm by 6.0 cm rotates one revolution about one of its longest sides as axis. Determine the volume of the resulting cylinder by using the theorem of Pappus. [1131 cm^2]

3. Using (a) the theorem of Pappus, and (b) integration, determine the position of the centroid of a metal template in the form of a quadrant of a circle of radius 4 cm. (The equation of a circle, centre 0, radius r is $x^2 + y^2 = r^2$).
[On the centre line, distance 2.40 cm from the centre, i.e. at coordinates (1.70, 1.70)]

4. (a) Determine the area bounded by the curve $y = 5x^2$, the x-axis and the ordinates $x = 0$ and $x = 3$.
   (b) If this area is revolved 360° about (i) the x-axis, and (ii) the y-axis, find the volumes of the solids of revolution produced in each case.
   (c) Determine the co-ordinates of the centroid of the area using (i) integral calculus, and (ii) the theorem of Pappus

   [(a) 45 square units (b) (i) 1215π cubic units (ii) 202.5π cubic units (c) (2.25, 13.5)]

5. A metal disc has a radius of 7.0 cm and is of thickness 2.5 cm. A semicircular groove of diameter 2.0 cm is machined centrally around the rim to form a pulley. Determine the volume of metal removed using Pappus' theorem and express this as a percentage of the original volume of the disc. Find also the mass of metal removed if the density of the metal is $7800\,kg\,m^{-3}$.

   [$64.90\,cm^3$, 16.86%, 506.2 g]

Chapter 59

Second moments of area

59.1 Second moments of area and radius of gyration

The **first moment of area** about a fixed axis of a lamina of area A, perpendicular distance y from the centroid of the lamina is defined as Ay cubic units. The **second moment of area** of the same lamina as above is given by Ay^2, i.e. the perpendicular distance from the centroid of the area to the fixed axis is squared. Second moments of areas are usually denoted by I and have limits of mm^4, cm^4, and so on.

Radius of gyration

Several areas, $a_1, a_2, a_3, \ldots$ at distances $y_1, y_2, y_3, \ldots$ from a fixed axis, may be replaced by a single area A, where $A = a_1 + a_2 + a_3 + \cdots$ at distance k from the axis, such that $Ak^2 = \sum ay^2$. k is called the **radius of gyration** of area A about the given axis. Since $Ak^2 = \sum ay^2 = I$ then the radius of gyration, $\boldsymbol{k = \sqrt{\frac{I}{A}}}$

The second moment of area is a quantity much used in the theory of bending of beams, in the torsion of shafts, and in calculations involving water planes and centres of pressure.

59.2 Second moment of area of regular sections

The procedure to determine the second moment of area of regular sections about a given axis is (i) to find the second moment of area of a typical element and (ii) to sum all such second moments of area by integrating between appropriate limits.

For example, the second moment of area of the rectangle shown in Fig. 59.1 about axis PP is found by initially considering an elemental strip of width δx, parallel to and distance x from axis PP. Area of shaded strip $= b\delta x$. Second moment of area of the shaded strip about $PP = (x^2)(b\delta x)$.

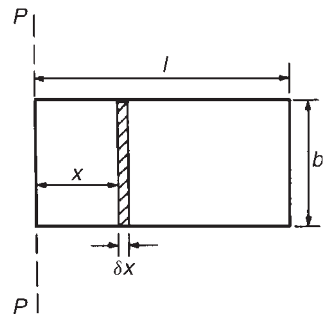

Figure 59.1

The second moment of area of the whole rectangle about PP is obtained by summing all such strips between $x = 0$ and $x = l$, i.e. $\sum_{x=0}^{x=l} x^2 b\delta x$. It is a fundamental theorem of integration that

$$\underset{\delta x \to 0}{\text{limit}} \sum_{x=0}^{x=l} x^2 b\delta x = \int_0^l x^2 b\, dx$$

Thus the second moment of area of the rectangle about

$$PP = b\int_0^l x^2 dx = b\left[\frac{x^3}{3}\right]_0^l = \boldsymbol{\frac{bl^3}{3}}$$

Since the total area of the rectangle, $A = lb$, then

$$I_{pp} = (lb)\left(\frac{l^2}{3}\right) = \boldsymbol{\frac{Al^2}{3}}$$

$$I_{pp} = Ak_{pp}^2 \text{ thus } k_{pp}^2 = \frac{l^2}{3}$$

i.e. the radius of gyration about axes PP,

$$\boldsymbol{k_{pp}} = \sqrt{\frac{l^2}{3}} = \boldsymbol{\frac{l}{\sqrt{3}}}$$

59.3 Parallel axis theorem

In Fig. 59.2, axis GG passes through the centroid C of area A. Axes DD and GG are in the same plane, are parallel to each other and distance d apart. The parallel axis theorem states:

$$\mathbf{I_{DD} = I_{GG} + Ad^2}$$

Using the parallel axis theorem the second moment of area of a rectangle about an axis through the centroid may be determined. In the rectangle shown in Fig. 59.3, $I_{pp} = \frac{bl^3}{3}$ (from above). From the parallel axis theorem

$$I_{pp} = I_{GG} + (bl)\left(\frac{l}{2}\right)^2$$

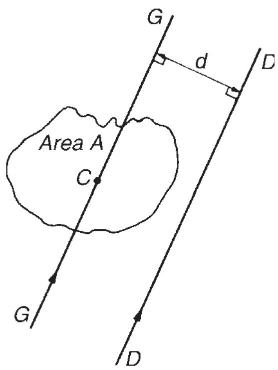

Figure 59.2

i.e. $$\frac{bl^3}{3} = I_{GG} + \frac{bl^3}{4}$$

from which, $$\mathbf{I_{GG}} = \frac{bl^3}{3} - \frac{bl^3}{4} = \mathbf{\frac{bl^3}{12}}$$

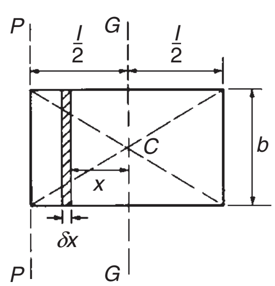

Figure 59.3

59.4 Perpendicular axis theorem

In Fig. 59.4, axes OX, OY and OZ are mutually perpendicular. If OX and OY lie in the plane of area A then the perpendicular axis theorem states:

$$\mathbf{I_{OZ} = I_{OX} + I_{OY}}$$

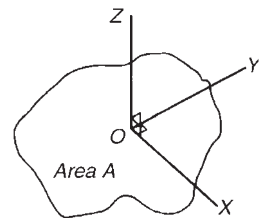

Figure 59.4

59.5 Summary of derived results

A summary of derive standard results for the second moment of area and radius of gyration of regular sections are listed in Table 59.1.

Table 59.1 Summary of standard results of the second moments of areas of regular sections

Shape	Position of axis	Second moment of area, I	Radius of gyration, k
Rectangle length l breadth b	(1) Coinciding with b	$\frac{bl^3}{3}$	$\frac{l}{\sqrt{3}}$
	(2) Coinciding with l	$\frac{lb^3}{3}$	$\frac{b}{\sqrt{3}}$
	(3) Through centroid, parallel to b	$\frac{bl^3}{12}$	$\frac{l}{\sqrt{12}}$
	(4) Through centroid, parallel to l	$\frac{lb^3}{12}$	$\frac{b}{\sqrt{12}}$
Triangle Perpendicular height h base b	(1) Coinciding with b	$\frac{bh^3}{12}$	$\frac{h}{\sqrt{6}}$
	(2) Through centroid, parallel to base	$\frac{bh^3}{36}$	$\frac{h}{\sqrt{18}}$
	(3) Through vertex, parallel to base	$\frac{bh^3}{4}$	$\frac{h}{\sqrt{2}}$
Circle radius r	(1) Through centre, perpendicular to plane (i.e. polar axis)	$\frac{\pi r^4}{2}$	$\frac{r}{\sqrt{2}}$
	(2) Coinciding with diameter	$\frac{\pi r^4}{4}$	$\frac{r}{2}$
	(3) About a tangent	$\frac{5\pi r^4}{4}$	$\frac{\sqrt{5}}{2}r$
Semicircle radius r	Coinciding with diameter	$\frac{\pi r^4}{8}$	$\frac{r}{2}$

59.6 Worked problems on second moments of area of regular sections

Problem 1. Determine the second moment of area and the radius of gyration about axes AA, BB and CC for the rectangle shown in Fig. 59.5.

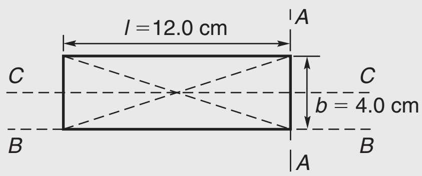

Figure 59.5

From Table 59.1, the second moment of area about axis AA, $I_{AA} = \frac{bl^3}{3} = \frac{(4.0)(12.0)^3}{3} = \mathbf{2304\ cm^4}$

Radius of gyration,

$$k_{AA} = \frac{l}{\sqrt{3}} = \frac{12.0}{\sqrt{3}} = \mathbf{6.93\ cm}$$

Similarly, $$I_{BB} = \frac{lb^3}{3} = \frac{(12.0)(4.0)^3}{3} = \mathbf{256\ cm^4}$$

and $$k_{BB} = \frac{b}{\sqrt{3}} = \frac{4.0}{\sqrt{3}} = \mathbf{2.31\ cm}$$

The second moment of area about the centroid of a rectangle is $\frac{bl^3}{12}$ when the axis through the centroid is parallel with the breadth, b. In this case, the axis CC is parallel with the length l.

Hence $$I_{CC} = \frac{lb^3}{12} = \frac{(12.0)(4.0)^3}{12} = \mathbf{64\ cm^4}$$

and $$k_{CC} = \frac{b}{\sqrt{12}} = \frac{4.0}{\sqrt{12}} = \mathbf{1.15\ cm}$$

Problem 2. Find the second moment of area and the radius of gyration about axis PP for the rectangle shown in Fig. 59.6

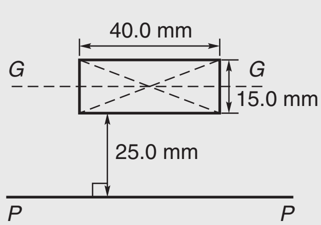

Figure 59.6

$$I_{GG} = \frac{lb^3}{12} \text{ where } l = 40.0\ \text{mm} \quad \text{and} \quad b = 15.0\ \text{mm}$$

Hence $I_{GG} = \frac{(40.0)(15.0)^3}{12} = 11\,250\ \text{mm}^4$

From the parallel axis theorem, $I_{PP} = I_{GG} + Ad^2$, where $A = 40.0 \times 15.0 = 600\ \text{mm}^2$ and $d = 25.0 + 7.5 = 32.5\ \text{mm}$, the perpendicular distance between GG and PP.

Hence, $$I_{PP} = 11\,250 + (600)(32.5)^2 = \mathbf{645\,000\ mm^4}$$

$$I_{PP} = Ak_{PP}^2$$

from which, $$k_{PP} = \sqrt{\frac{I_{PP}}{\text{area}}} = \sqrt{\frac{645\,000}{600}} = \mathbf{32.79\ mm}$$

Problem 3. Determine the second moment of area and radius of gyration about axis QQ of the triangle BCD shown in Fig. 59.7

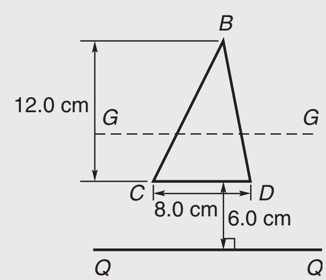

Figure 59.7

Using the parallel axis theorem: $I_{QQ} = I_{GG} + Ad^2$, where I_{GG} is the second moment of area about the centroid of the triangle,

i.e. $\frac{bh^3}{36} = \frac{(8.0)(12.0)^3}{36} = 384\ \text{cm}^4$, A is the area of the triangle $= \frac{1}{2}bh = \frac{1}{2}(8.0)(12.0) = 48\ \text{cm}^2$ and d is the distance between axes GG and $QQ = 6.0 + \frac{1}{3}(12.0) = 10\ \text{cm}$.

Hence the second moment of area about axis QQ,

$$I_{QQ} = 384 + (48)(10)^2 = \mathbf{5184\ cm^4}$$

Radius of gyration,

$$k_{QQ} = \sqrt{\frac{I_{QQ}}{\text{area}}} = \sqrt{\frac{5184}{48}} = \mathbf{10.4\ cm}$$

Problem 4. Determine the second moment of area and radius of gyration of the circle shown in Fig. 59.8 about axis YY

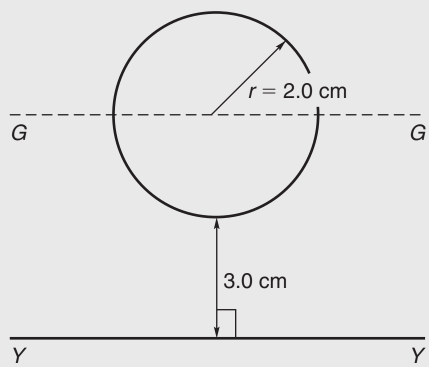

Figure 59.8

In Fig. 59.8, $I_{GG} = \frac{\pi r^4}{4} = \frac{\pi}{4}(2.0)^4 = 4\pi \text{ cm}^4$. Using the parallel axis theorem, $I_{YY} = I_{GG} + Ad^2$, where $d = 3.0 + 2.0 = 5.0$ cm.

Hence $\quad \boldsymbol{I_{YY}} = 4\pi + [\pi(2.0)^2](5.0)^2$

$$= 4\pi + 100\pi = 104\pi = \mathbf{327\ cm^4}$$

Radius of gyration,

$$\boldsymbol{k_{YY}} = \sqrt{\frac{I_{YY}}{\text{area}}} = \sqrt{\frac{104\pi}{\pi(2.0)^2}} = \sqrt{26} = \mathbf{5.10\ cm}$$

Problem 5. Determine the second moment of area and radius of gyration for the semicircle shown in Fig. 59.9 about axis XX

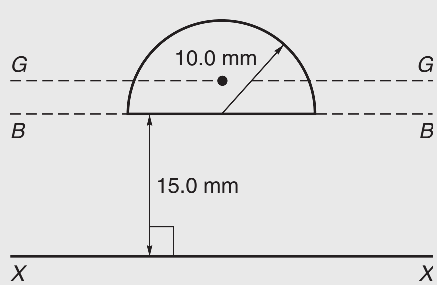

Figure 59.9

The centroid of a semicircle lies at $\frac{4r}{3\pi}$ from its diameter.

Using the parallel axis theorem: $I_{BB} = I_{GG} + Ad^2$,

where $\quad I_{BB} = \frac{\pi r^4}{8}$ (from Table 59.1)

$$= \frac{\pi(10.0)^4}{8} = 3927 \text{ mm}^4,$$

$$A = \frac{\pi r^2}{2} = \frac{\pi(10.0)^2}{2} = 157.1 \text{ mm}^2$$

and $\quad d = \frac{4r}{3\pi} = \frac{4(10.0)}{3\pi} = 4.244 \text{ mm}$

Hence $\quad 3927 = I_{GG} + (157.1)(4.244)^2$

i.e. $\quad 3927 = I_{GG} + 2830,$

from which, $\quad I_{GG} = 3927 - 2830 = 1097 \text{ mm}^4$

Using the parallel axis theorem again:

$I_{XX} = I_{GG} + A(15.0 + 4.244)^2$

i.e. $\boldsymbol{I_{XX}} = 1097 + (157.1)(19.244)^2$

$= 1097 + 58\,179 = 59\,276 \text{ mm}^4$ or $\mathbf{59\,280\ mm^4}$, correct to 4 significant figures.

Radius of gyration, $\boldsymbol{k_{XX}} = \sqrt{\frac{I_{XX}}{\text{area}}} = \sqrt{\frac{59\,276}{157.1}}$

$$= \mathbf{19.42\ mm}$$

Problem 6. Determine the polar second moment of area of the propeller shaft cross-section shown in Fig. 59.10

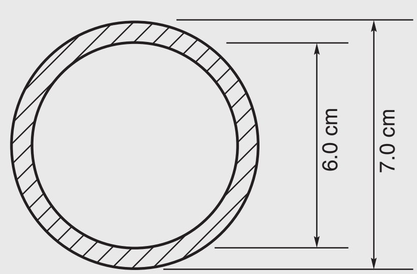

Figure 59.10

The polar second moment of area of a circle $= \frac{\pi r^4}{2}$.
The polar second moment of area of the shaded area is given by the polar second moment of area of the 7.0 cm diameter circle minus the polar second moment of area of the 6.0 cm diameter circle. Hence the polar second

moment of area of the

$$\text{cross-section shown} = \frac{\pi}{2}\left(\frac{7.0}{2}\right)^4 - \frac{\pi}{2}\left(\frac{6.0}{2}\right)^4$$

$$= 235.7 - 127.2 = \mathbf{108.5\ cm^4}$$

Problem 7. Determine the second moment of area and radius of gyration of a rectangular lamina of length 40 mm and width 15 mm about an axis through one corner, perpendicular to the plane of the lamina

The lamina is shown in Fig. 59.11.

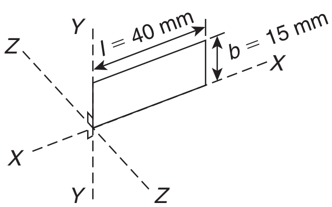

Figure 59.11

From the perpendicular axis theorem:

$$I_{ZZ} = I_{XX} + I_{YY}$$

$$I_{XX} = \frac{lb^3}{3} = \frac{(40)(15)^3}{3} = 45\,000\ \text{mm}^4$$

and $$I_{YY} = \frac{bl^3}{3} = \frac{(15)(40)^3}{3} = 320\,000\ \text{mm}^4$$

Hence $$\boldsymbol{I_{ZZ}} = 45\,000 + 320\,000$$

$$= \mathbf{365\,000\ mm^4} \text{ or } \mathbf{36.5\ cm^4}$$

Radius of gyration,

$$k_{ZZ} = \sqrt{\frac{I_{ZZ}}{\text{area}}} = \sqrt{\frac{365\,000}{(40)(15)}}$$

$$= \mathbf{24.7\ mm} \text{ or } \mathbf{2.47\ cm}$$

Now try the following exercise

Exercise 201 Further problems on second moments of area of regular sections

1. Determine the second moment of area and radius of gyration for the rectangle shown in Fig. 59.12 about (a) axis *AA* (b) axis *BB*, and (c) axis *CC*.

   [(a) 72 cm^4, 1.73 cm (b) 128 cm^4, 2.31 cm (c) 512 cm^4, 4.62 cm]

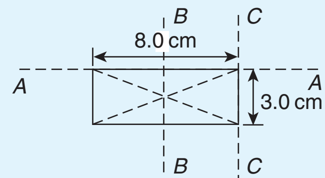

Figure 59.12

2. Determine the second moment of area and radius of gyration for the triangle shown in Fig. 59.13 about (a) axis *DD* (b) axis *EE*, and (c) an axis through the centroid of the triangle parallel to axis *DD*.

   [(a) 729 cm^4, 3.67 cm (b) 2187 cm^4, 6.36 cm (c) 243 cm^4, 2.12 cm]

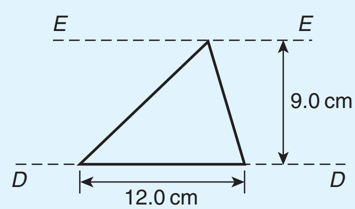

Figure 59.13

3. For the circle shown in Fig. 59.14, find the second moment of area and radius of gyration about (a) axis *FF*, and (b) axis *HH*.

   [(a) 201 cm^4, 2.0 cm (b) 1005 cm^4, 4.47 cm]

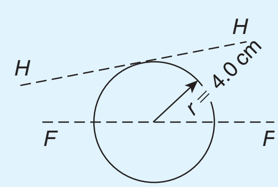

Figure 59.14

4. For the semicircle shown in Fig. 59.15, find the second moment of area and radius of gyration about axis *JJ*. [3927 mm^4, 5.0 mm]

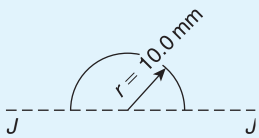

Figure 59.15

5. For each of the areas shown in Fig. 59.16 determine the second moment of area and radius of gyration about axis LL, by using the parallel axis theorem.
$$\left[\begin{array}{l}\text{(a) } 335\,\text{cm}^4,\ 4.73\,\text{cm}\\ \text{(b) } 22\,030\,\text{cm}^4,\ 14.3\,\text{cm}\\ \text{(c) } 628\,\text{cm}^4,\ 7.07\,\text{cm}\end{array}\right]$$

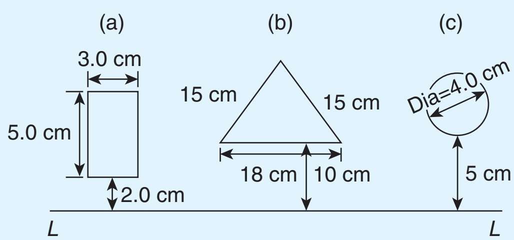

Figure 59.16

6. Calculate the radius of gyration of a rectangular door 2.0 m high by 1.5 m wide about a vertical axis through its hinge. [0.866 m]

7. A circular door of a boiler is hinged so that it turns about a tangent. If its diameter is 1.0 m, determine its second moment of area and radius of gyration about the hinge. [0.245 m^4, 0.599 m]

8. A circular cover, centre 0, has a radius of 12.0 cm. A hole of radius 4.0 cm and centre X, where $OX = 6.0$ cm, is cut in the cover. Determine the second moment of area and the radius of gyration of the remainder about a diameter through 0 perpendicular to OX. [14 280 cm^4, 5.96 cm]

59.7 Worked problems on second moments of area of composite areas

Problem 8. Determine correct to 3 significant figures, the second moment of area about XX for the composite area shown in Fig. 59.17

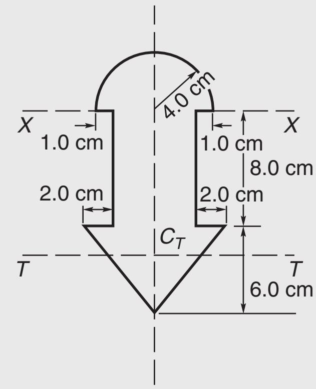

Figure 59.17

For the semicircle, $I_{XX} = \dfrac{\pi r^4}{8} = \dfrac{\pi(4.0)^4}{8}$

$= 100.5\,\text{cm}^4$

For the rectangle, $I_{XX} = \dfrac{bl^3}{3} = \dfrac{(6.0)(8.0)^3}{3}$

$= 1024\,\text{cm}^4$

For the triangle, about axis TT through centroid C_T,

$$I_{TT} = \frac{bh^3}{36} = \frac{(10)(6.0)^3}{36} = 60\,\text{cm}^4$$

By the parallel axis theorem, the second moment of area of the triangle about axis XX

$$= 60 + \left[\tfrac{1}{2}(10)(6.0)\right]\left[8.0 + \tfrac{1}{3}(6.0)\right]^2 = 3060\,\text{cm}^4.$$

Total second moment of area about XX.
$= 100.5 + 1024 + 3060 = 4184.5 = \mathbf{4180\,cm^4}$, correct to 3 significant figures

Problem 9. Determine the second moment of area and the radius of gyration about axis XX for the I-section shown in Fig. 59.18

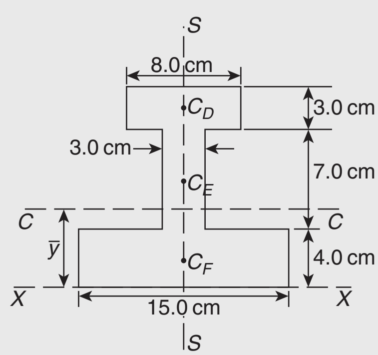

Figure 59.18

The I-section is divided into three rectangles, D, E and F and their centroids denoted by C_D, C_E and C_F respectively.

For rectangle D:
The second moment of area about C_D (an axis through C_D parallel to XX)

$$= \frac{bl^3}{12} = \frac{(8.0)(3.0)^3}{12} = 18\,\text{cm}^4$$

Using the parallel axis theorem: $I_{XX} = 18 + Ad^2$ where $A = (8.0)(3.0) = 24\,\text{cm}^2$ and $d = 12.5\,\text{cm}$

Hence $\boldsymbol{I_{XX}} = 18 + 24(12.5)^2 = \mathbf{3768\,cm^4}$

For rectangle E:
The second moment of area about C_E (an axis through C_E parallel to XX)

$$= \frac{bl^3}{12} = \frac{(3.0)(7.0)^3}{12} = 85.75\,\text{cm}^4$$

Using the parallel axis theorem:
$\boldsymbol{I_{XX}} = 85.75 + (7.0)(3.0)(7.5)^2 = \mathbf{1267\,cm^4}$

For rectangle F:

$$\boldsymbol{I_{XX}} = \frac{bl^3}{3} = \frac{(15.0)(4.0)^3}{3} = \mathbf{320\,cm^4}$$

Total second moment of area for the *I*-section about axis *XX*,

$$\boldsymbol{I_{XX}} = 3768 + 1267 + 320 = \mathbf{5355\,cm^4}$$

Total area of I-section

$$= (8.0)(3.0) + (3.0)(7.0) + (15.0)(4.0) = 105\,\text{cm}^2.$$

Radius of gyration,

$$\boldsymbol{k_{XX}} = \sqrt{\frac{I_{XX}}{\text{area}}} = \sqrt{\frac{5355}{105}} = \mathbf{7.14\,cm}$$

Now try the following exercise

Exercise 202 Further problems on section moment of areas of composite areas

1. For the sections shown in Fig. 59.19, find the second moment of area and the radius of gyration about axis XX.
   [(a) $12\,190\,\text{mm}^4$, 10.9 mm; (b) $549.5\,\text{cm}^4$, 4.18 cm]

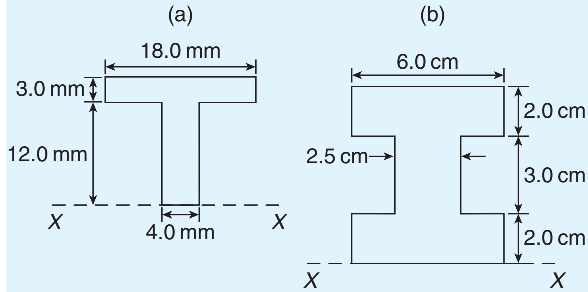

Figure 59.19

2. Determine the second moment of area about the given axes for the shapes shown in Fig. 59.20. (In Fig. 59.20(b), the circular area is removed.)
   [$I_{AA} = 4224\,\text{cm}^4$, $I_{BB} = 6718\,\text{cm}^4$, $I_{CC} = 37\,300\,\text{cm}^4$]

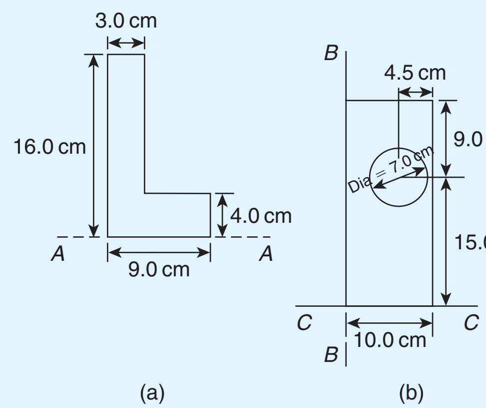

Figure 59.20

3. Find the second moment of area and radius of gyration about the axis XX for the beam section shown in Fig. 59.21. [$1350\,\text{cm}^4$, 5.67 cm]

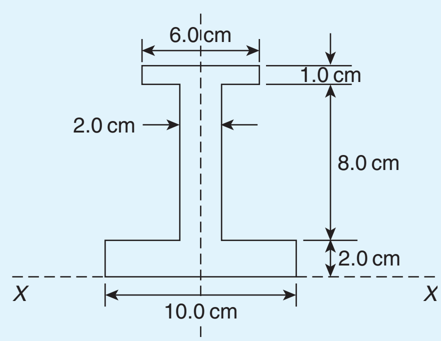

Figure 59.21

Revision Test 16

This Revision test covers the material contained in Chapters 55 to 59. *The marks for each question are shown in brackets at the end of each question.*

1. The force F newtons acting on a body at a distance x metres from a fixed point is given by: $F = 2x + 3x^2$. If work done $= \int_{x_1}^{x_2} F\,dx$, determine the work done when the body moves from the position when $x = 1$ m to that when $x = 4$ m. (4)

2. Sketch and determine the area enclosed by the curve $y = 3\sin\frac{\theta}{2}$, the θ-axis and ordinates $\theta = 0$ and $\theta = \frac{2\pi}{3}$. (4)

3. Calculate the area between the curve $y = x^3 - x^2 - 6x$ and the x-axis. (10)

4. A voltage $v = 25\sin 50\pi t$ volts is applied across an electrical circuit. Determine, using integration, its mean and r.m.s. values over the range $t = 0$ to $t = 20$ ms, each correct to 4 significant figures. (12)

5. Sketch on the same axes the curves $x^2 = 2y$ and $y^2 = 16x$ and determine the co-ordinates of the points of intersection. Determine (a) the area enclosed by the curves, and (b) the volume of the solid produced if the area is rotated one revolution about the x-axis. (13)

6. Calculate the position of the centroid of the sheet of metal formed by the x-axis and the part of the curve $y = 5x - x^2$ which lies above the x-axis. (9)

7. A cylindrical pillar of diameter 500 mm has a groove cut around its circumference as shown in Fig. R16.1. The section of the groove is a semicircle of diameter 40 mm. Given that the centroid of a semicircle from its base is $\frac{4r}{3\pi}$, use the theorem

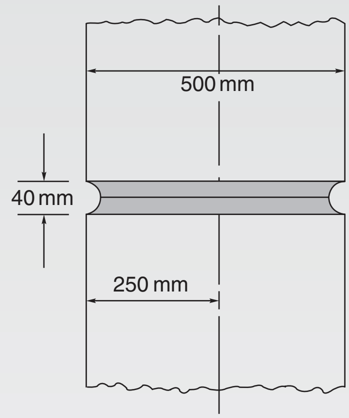

Figure R16.1

of Pappus to determine the volume of material removed, in cm^3, correct to 3 significant figures. (8)

8. For each of the areas shown in Fig. R16.2 determine the second moment of area and radius of gyration about axis XX. (15)

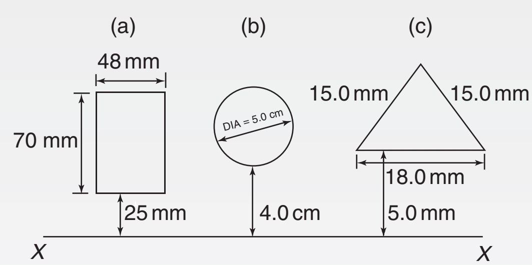

Figure R16.2

9. A circular door is hinged so that it turns about a tangent. If its diameter is 1.0 m find its second moment of area and radius of gyration about the hinge. (5)