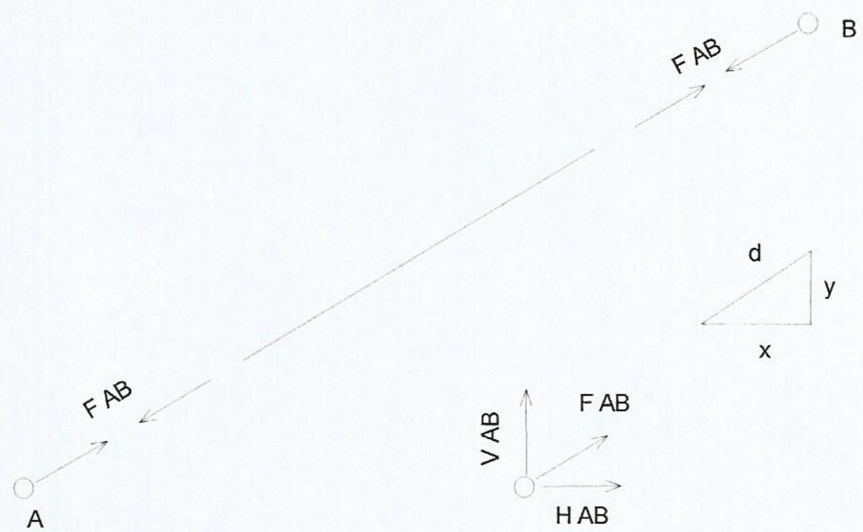
TRUSSES

Method of Joints

$$\sum F_x = 0 \qquad \sum F_y = 0$$



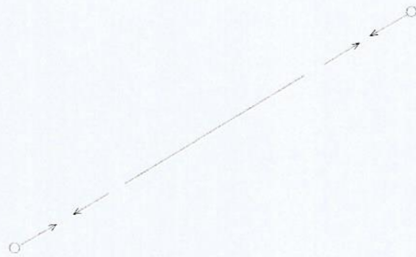
$$\text{Force in Required Direction} = \frac{\textit{Given Force}}{\textit{Given Force Dist.}} * \textit{Required Force Distance}$$

If H_{AB} is given:

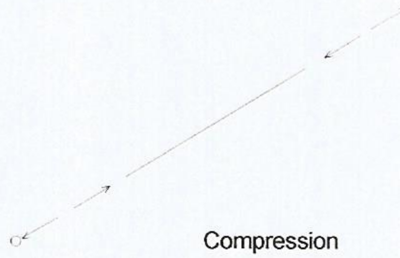
$$V_{AB} = \frac{H_{AB}}{x} * y \qquad F_{AB} = \frac{H_{AB}}{x} * d$$

If V_{AB} is given:

$$H_{AB} = \frac{V_{AB}}{y} * x \qquad F_{AB} = \frac{V_{AB}}{y} * d$$



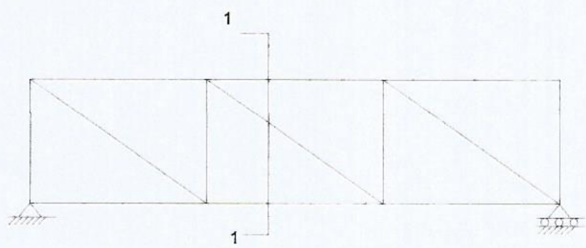
Tension

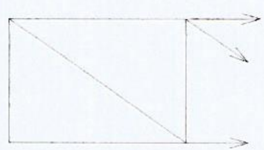


Compression

Method of Sections

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M = 0$$



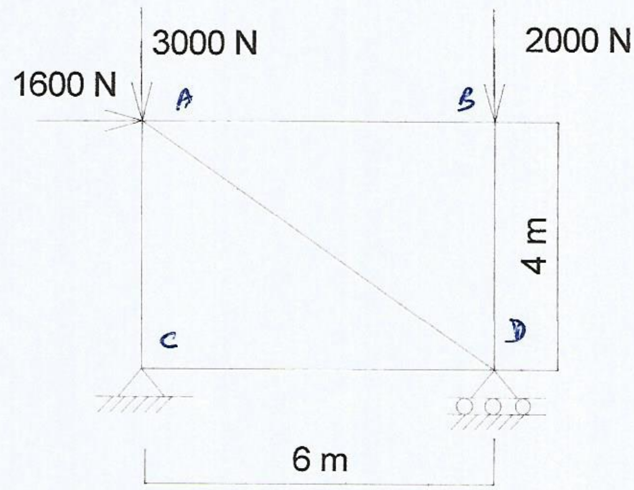


Section 1-1

LHS

Example

For the truss shown, use the joints method to determine all member forces.



Section 1-1

Solution

$\sum M_C = 0$

$1600*4+2000*6-D_y*6=0 \qquad D_y = 3066.7\,N$ ↑

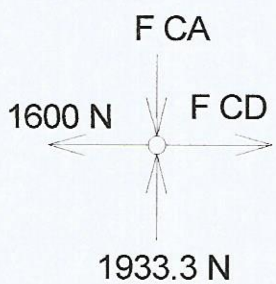
$\sum F_y = 0$

$-2000-3000+3066.7+C_y=0 \qquad C_y = 1933.3\,N$ ↑

$\sum F_x = 0$

$1600-C_x=0 \qquad C_x = 1600\,N$ ←

Joint C



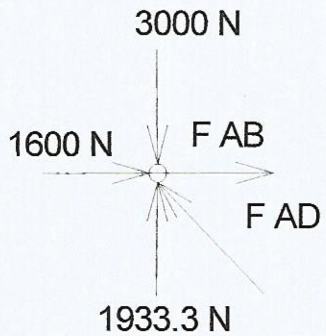
$\sum F_y = 0$

$1933.3-F_{CA}=0 \qquad F_{CA} = 1933.3\,N\,(Comp.)$

$\sum F_x = 0$

$F_{CD}-1600=0 \qquad F_{CD} = 1600\,N\,(Tens.)$

Joint A



$$\sum F_y = 0$$

$$-3000 + 1933.3 - V_{AD} = 0 \qquad V_{AD} = 1066.7.3\,N\uparrow$$

$$F_{AD} = 1066.7 * \frac{7.21}{4} = 1922.7N(Comp.)$$

$$H_{AD} = 1066.7 * \frac{6}{4} = 1600N \rightarrow$$

$$\sum F_x = 0$$

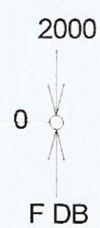
$$F_{AB} - 1600 + 1600 = 0 \qquad F_{AB} = 0$$

Joint B

$$\sum F_y = 0$$

$$-2000 + F_{DB} = 0 \qquad F_{DB} = 2000\,N\,(Comp.)$$

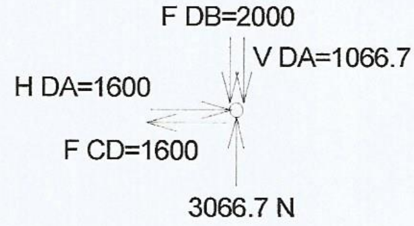
$$\sum F_x = 0 \qquad 0 = 0 \quad OK$$



Checking at Joint D

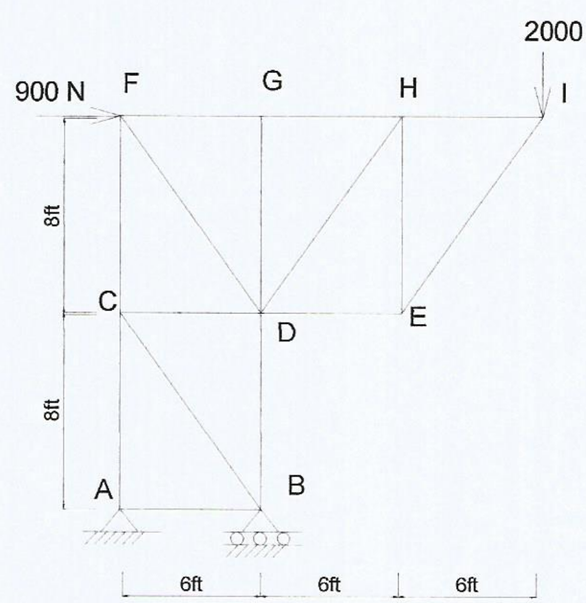
$$\sum Fx = 0 \qquad 1600 - 1600 = 0 \quad OK$$

$$\sum Fy = 0 \qquad 3066.7 - 2000 - 1066.7 = 0 \quad OK$$



Example

For the truss shown in the figure, determine forces in members BC, FG, GH and HI.



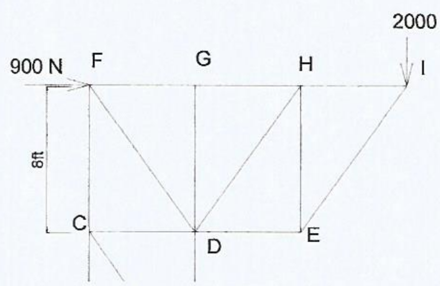
Solution

Section 1-1 upper

$$\sum F_x = 0$$

$$-H_{CB} + 900 = 0 \qquad H_{CB} = 900$$

$$F_{CB} = 900 * \frac{10}{6} = 1500\,Ib$$ (Comp.)

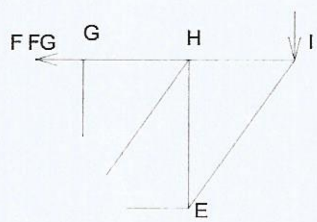


Section 2-2 Right

$$\sum M_D = 0$$

$$2000 * 12 - F_{FG} * 8 = 0$$

$$F_{FG} = 3000\,Ib \quad (Tens.)$$



Joint G

$$\sum Fy = 0 \qquad F_{GD} = 0$$

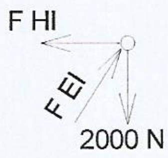
Joint I

$\sum F_y = 0$

$V_{EI} = 2000\ Ib$

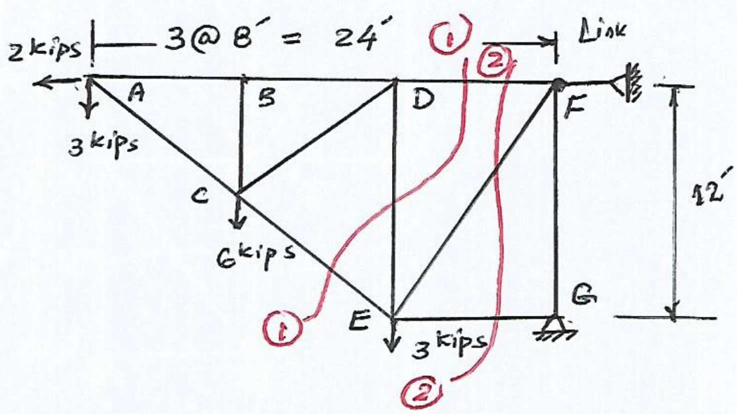
$H_{EI} = 2000 * \frac{6}{8} = 1500\ Ib$

$\sum Fx = 0 \qquad F_{HI} = 1500\ Ib\,(Comp.)$



Example

For the truss shown, determine the forces in members BC, DE and EF.



Solution

Joint B

$\sum Fy = 0 \ \uparrow$

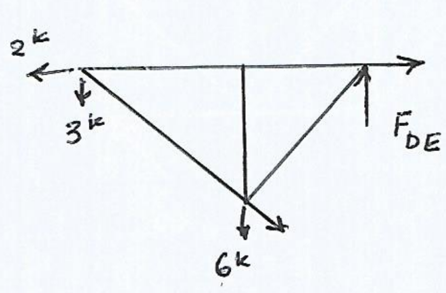
$F_{CB} = 0$

Section 1-1 Left

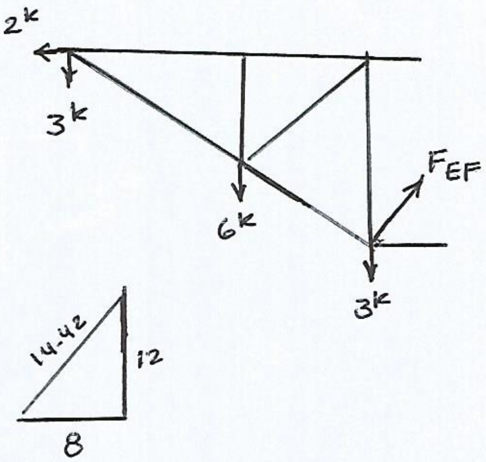
$\sum M_A = 0$

$6 * 8 - F_{DE} * 16 = 0$

$F_{DE} = 3\ kips \quad (Comp.)$



Section 2-2 Left



$\sum Fy = 0 \uparrow$

$V_{EF} - 3 - 6 - 3 = 0$

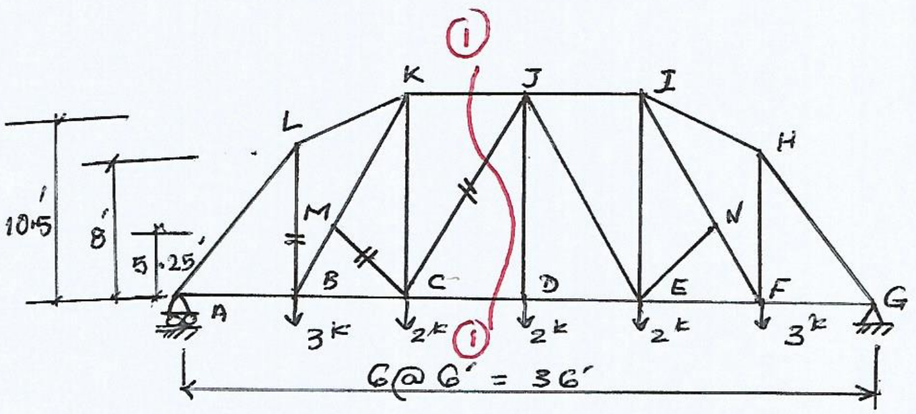
$V_{EF} = 12$

$F_{EF} = 12 * \frac{14.42}{12}$

$= 14.42\ kips \quad (Tens.)$

Example

Determine member forces of MC, CJ and BL of the shown truss.



Solution

The whole structure

$\sum M_G = 0$

$Y_B * 36 - 3 * 30 - 3 * 34 - 2 * 18 - 2 * 12 - 3 * 6 = 0$

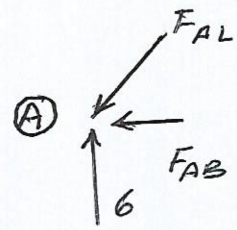
$Y_B = 6\ k$

$\sum Fy = 0 \uparrow$

$Y_B - 3 - 2 + 6 - 2 - 2 - 3 = 0$

$Y_G = 6\ k$

Joint A



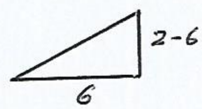
$\sum Fy = 0 \uparrow$

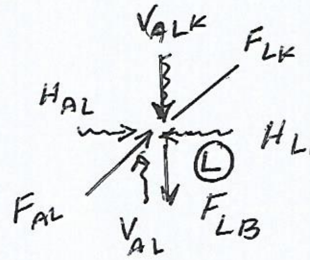
$V_{AL} = 6\ k$

$H_{AL} = 6 * \dfrac{6}{8} = 4.5\ k$

$\sum Fx = 0 \quad F_{AB} = 4.5\ k\ (Tens.)$

Joint L





$\sum Fx = 0$

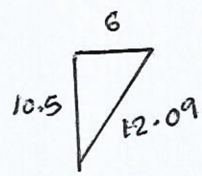
$H_{LK} = 4.5\ k \leftarrow$

$V_{LK} = 4.5 * \dfrac{2.5}{6} = 1.875\ k \downarrow$

$\sum Fy = 0 \uparrow$

$6 - 1.875 - F_{LB} = 0 \quad F_{LB} = 4.125\ k\ (Tens.)$

Section 1-1 Left



$\sum Fy = 0 \uparrow$

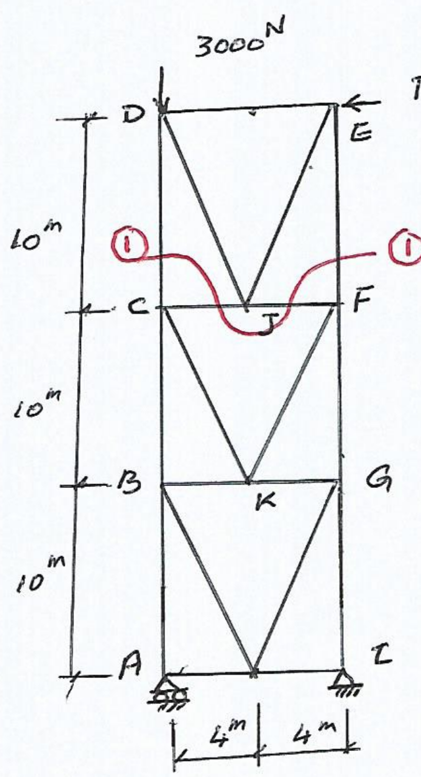
$6 - 3 - 2 - V_{CJ} = 0 \quad V_{CJ} = 1\ k \uparrow$

$F_{CJ} = 1 * \dfrac{12.09}{10.5}$

$= 1.15\ k\ (Comp.)$

Example

The force in member AB of the k-truss shown in the figure is 6000 N compression. Determine the force P and force in member EF.

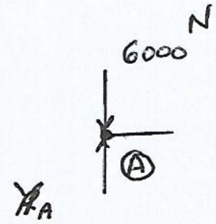


Solution

Joint A

$\sum Fy = 0$

$Y_A = 6000\ N \uparrow$



The whole structure

$\sum M_I = 0$

$-3000*8 - P*30 + 6000*8 = 0$

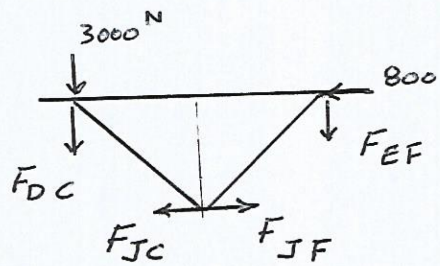
$P = 800\ N$

Section 1-1 Upper

$\sum M_C = 0$

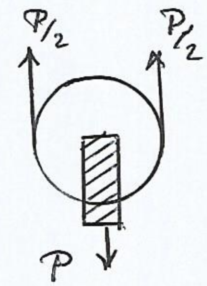
$-800*10 + F_{EF}*8 = 0$

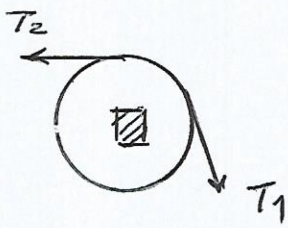
$F_{EF} = 1000\ N\ (Tension)$



PULLEYS

Pulley without a moment



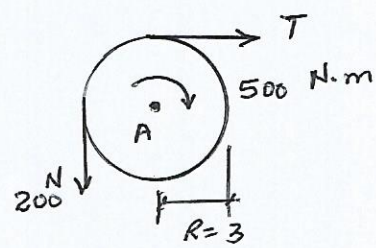


$\sum M_{Center}$

$T1*r - T2*r = 0$

$T1 = T2$

Pulley with a moment



$\sum M_{Center}$

$T1*r - T2*r + M = 0$

$T1 \neq T2$

Example

Determine the tension in the cable (T) for the pulley shown

Solution

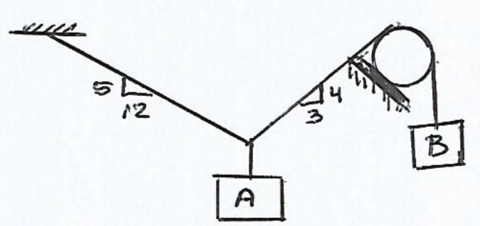
$\sum M_{A}$

$T1*3 - 200*3 + 500 = 0$

$T1 = 33.3\ N$

Example

For the cable system shown in the figure, if the weight of mass B is 2000^{N}, find the weight of mass A.



Solution

$\sum Fx = 0 \quad \rightarrow$

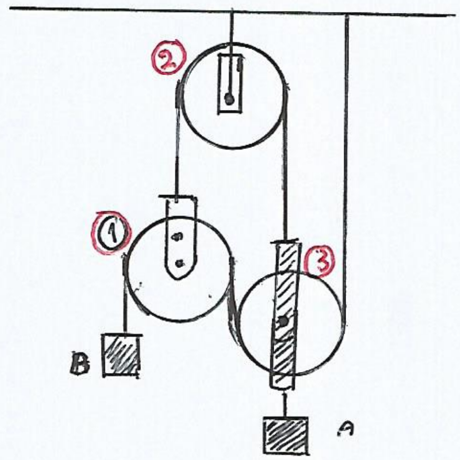
$$2000 * \frac{3}{5} - T_2 * \frac{12}{13} = 0$$

$$T_2 = 1300\ N$$

$\sum Fy = 0 \quad \uparrow$

$$1300 * \frac{5}{13} + 2000 * \frac{4}{5} - W_A = 0$$

$$W_A = 2100\ N$$



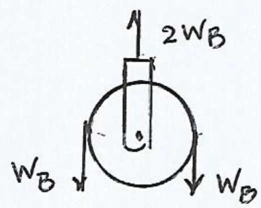
Example

For the system of pulleys shown

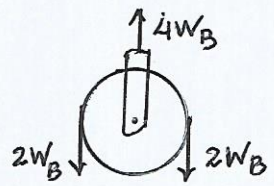
In the figure, if the weight of mass

A is 224^{Ib}. Find the weight of mass B.

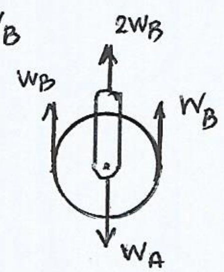
Pulley 1



Pulley 2



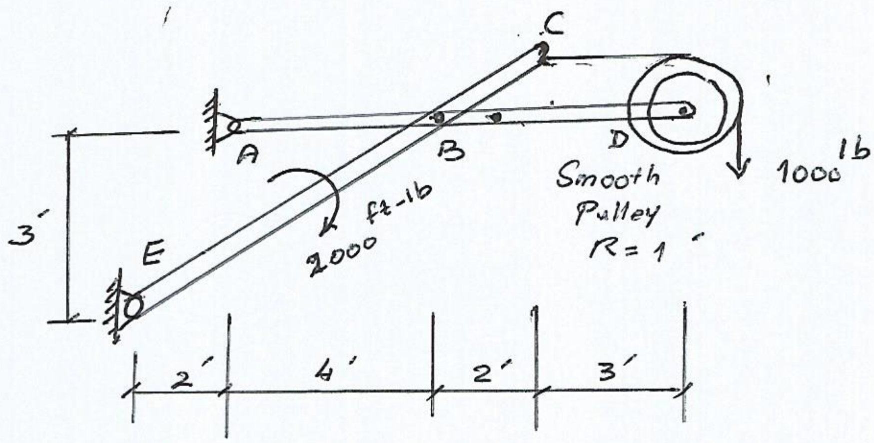
Pulley 3



$4W_B = 226 \qquad W_B = 56\ Ib$

Example

For the frame shown, determine pin reactions at B on member EBC



Solution

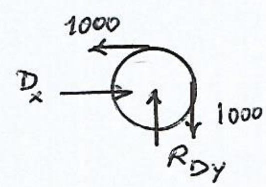
Pulley D as a F.B.D

$\sum Fx = 0 \quad \rightarrow$

$X_D - 1000 = 0 \quad X_D = 1000\ Ib \rightarrow$

$\sum Fy = 0 \quad \uparrow$

$Y_D - 1000 = 0 \quad X_D = 1000\ Ib \uparrow$



ABD as a F.B.D

$\sum M_A = 0$

$1000 * 9 - Y_B * 4 = 0$

$Y_B = 2250\ Ib \uparrow$

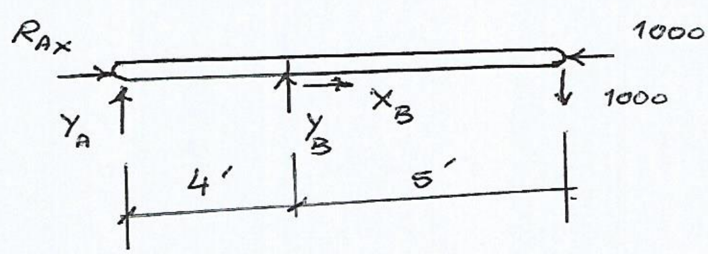
ABD as a F.B.D

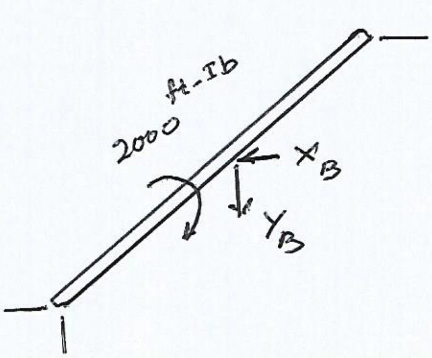
$\sum M_E = 0$

$2250 * 6 + 1000 * 4 - X_B * 3 = 0$

$X_B = 6500\ Ib \leftarrow$

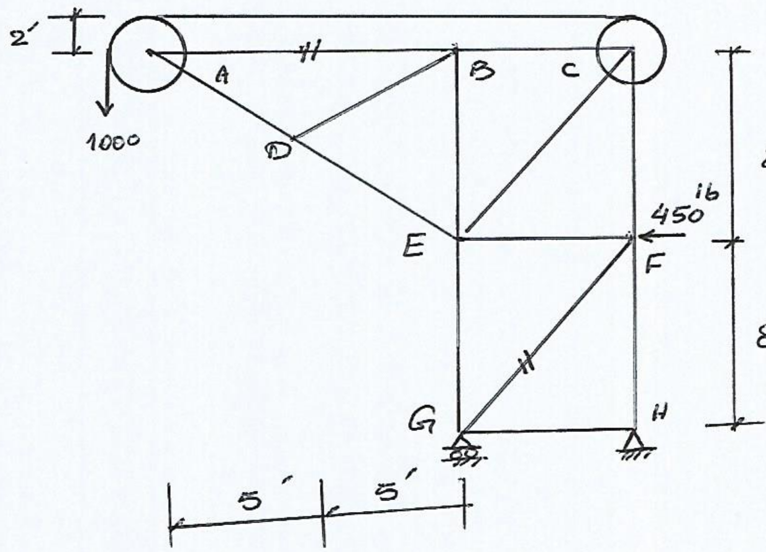
$Y_B = 2250\ Ib \downarrow$





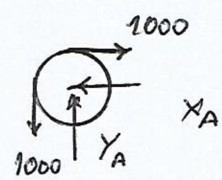
Example

Determine forces in members AB, EG, and GF



Solution

Pulley A



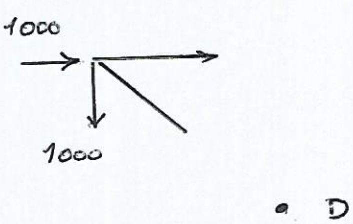
$\sum Fy = 0 \quad \uparrow$

$Y_A - 1000 = 0 \quad Y_A = 1000\ Ib \uparrow$

$\sum Fx = 0 \quad \rightarrow$

$X_A - 1000 = 0 \quad X_A = 1000\ Ib \leftarrow$

Section 1-1 Left

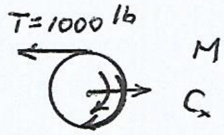


$\sum M_D = 0$

$-1000 * 5 + 1000 * 4 + F_{AB} * 4 = 0$

$F_{AB} = 250\ Ib\ (Tension)$

Pulley C



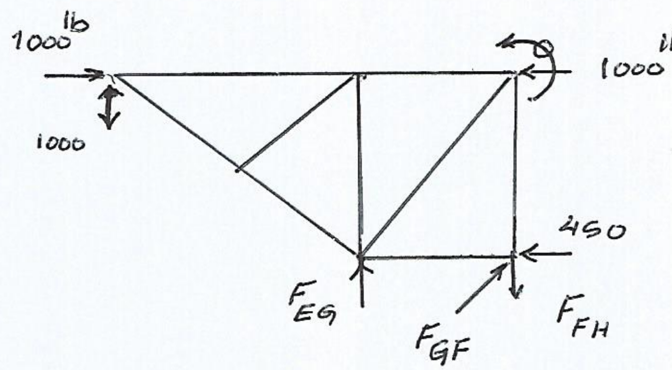
$\sum M_C = 0$

$M - 1000 * 2 = 0 \quad M = 2000\ ft - Ib$

$\sum Fx = 0 \quad \rightarrow$

$X_C - 1000 = 0 \quad X_C = 1000\ Ib$

Section 2-2 Upper



$\sum M_F = 0$

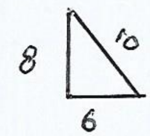
$1000*6 - 1000*6 - 1000*16 + F_{EG}*6 = 0$

$F_{EG} = 2666.7\ k\ (Comp.)$

$\sum Fx = 0 \quad \rightarrow$

$H_{GF} = 450\ Ib$

$F_{GF} = 450*\frac{10}{8} = 750\ Ib\ (Comp.)$



Joint A

$\sum Fy = 0$

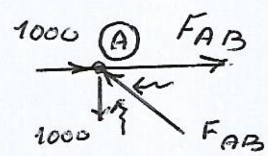
$F_{AD} = 1000\ Ib \uparrow$

$H_{AD} = 1000*\frac{5}{4} = 1250\ Ib \quad \leftarrow$

$\sum Fx = 0 \quad \rightarrow$

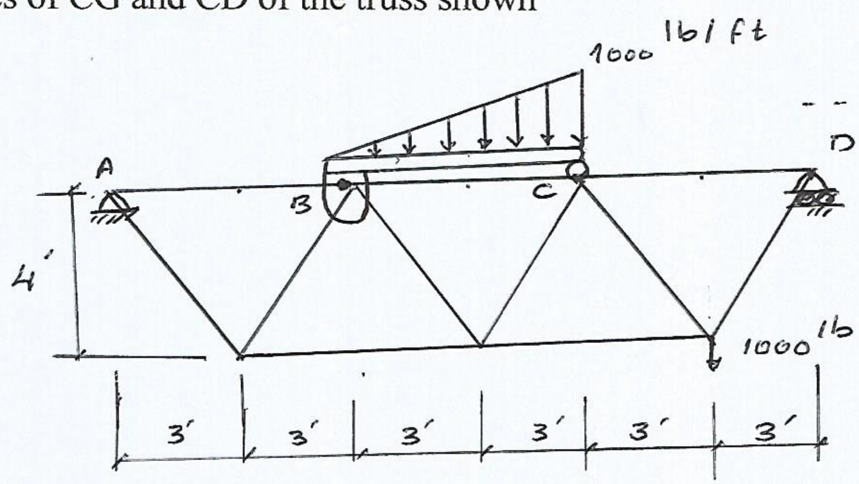
$F_{AB} + 1000 - 1250 = 0$

$F_{AB} = 250\ Ib\ (Tens.)$



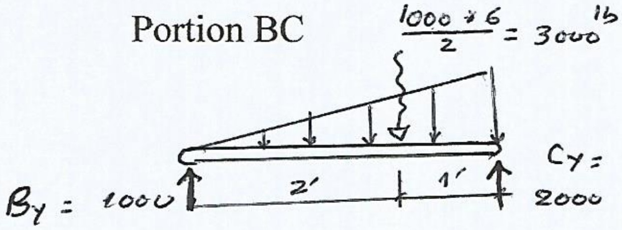
Example

Determine member forces of CG and CD of the truss shown



Solution

Portion BC



$\sum M_B = 0$

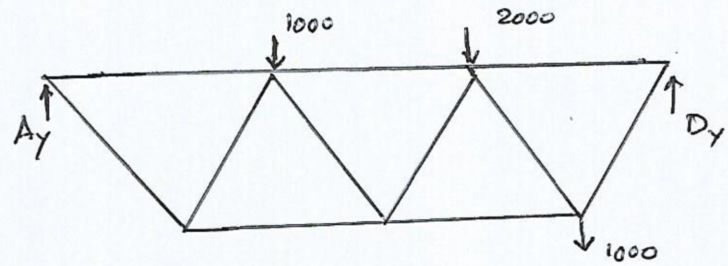
$\frac{1000*6}{2}*2 - C_y = 0$

$C_y = 2000\ Ib \uparrow$

$\sum Fy = 0 \quad \uparrow$

$-3000 + 2000 + B_y = 0$

$B_y = 1000\ Ib \uparrow$

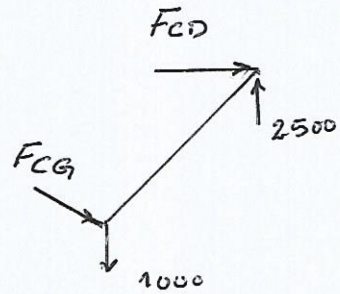


$\sum M_A = 0$

$1000*6 + 2000*12 + 1000*15 - Y_D*18 = 0$

$Y_D = 2500\ Ib \quad \uparrow$

Section 1-1 Right



$\sum M_G = 0$

$-2500*3 - F_{CD}*4 = 0$

$F_{CD} = 1875\ Ib\ (Comp.)$

$\sum Fy = 0$

$V_{CG} = 1500\ Ib \downarrow$

$F_{CG} = 1500*\frac{5}{4} = 1875\ Ib\ (Comp.)$

FRICTION

$f = \mu N$ where

f : frictional force (parallel to the surface)

N : Normal force (perpendicular to the surface)

μ : coefficient of friction

Depends on roughness of the two contacting surfaces

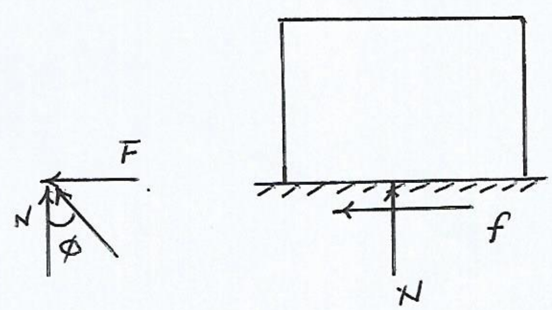
When the body is just starting to move :

$f = F$

$F = \mu N$

$\mu = \tan\phi$

ϕ : *Angle of friction*



If $f < F$ No motion (Equilibrium condition)

$f > F$ Motion

$f = F$ Body just starts to move.

The following statements mean that the body is about to move, i.e

$f = F = \mu N$

- Body just starts to move.
- Body to impend motion.
- Find maximum (or minimum) force for equilibrium of a body.
- Find the range of values of a force for equilibrium.
- Find minimum coefficient of friction.

Notes:

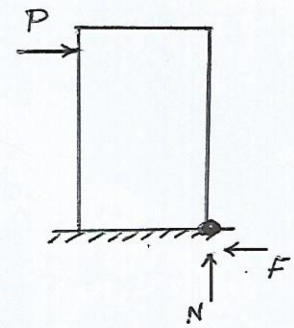
- If there is no dimensions given for the body and force locations, we only check :

Body sliding : $\sum Fx = 0 \qquad \sum Fy = 0$

- If the dimensions for the body and force locations are given, we must check :
  1. Body sliding : $\sum Fx = 0 \qquad \sum Fy = 0$
  2. Body tipping over (Overturn) :

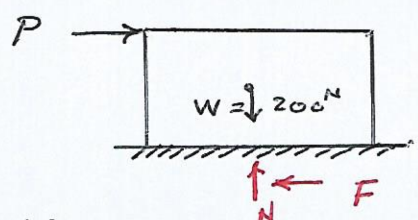
     $\sum Fx = 0 \qquad \sum Fy = 0$

     $\sum M \text{ about the expected point of overturn } = 0$



<u>*Example*</u>

Determine the maximum value of the force P before the body in the figure shown starts to move. Weight of the body is 200 N and coefficient of friction is 0.35



<u>*Solution*</u>

No dimensions are given for the body and force locations – check body sliding only.

Body slides to the right

$\sum Fy = 0$

$N = W = 200N$

$f = F = \mu N$

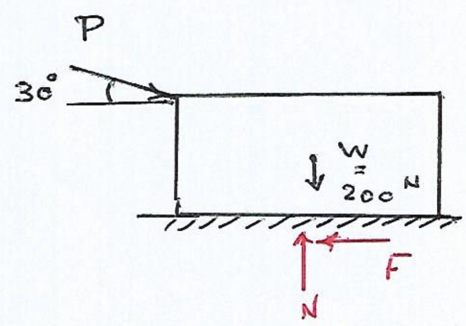
$= 0.35 * 200 = 70N \leftarrow$

$\sum Fx = 0$

$P\max = 70N \rightarrow$

<u>*Example*</u>

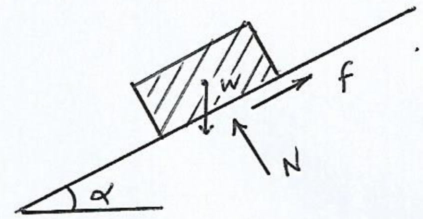
If the force P in the last example acts as shown in the figure, determine P max.



For a body sliding under the effect of its weight:

$N = W\cos\alpha$

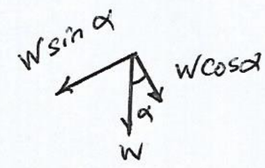
$f = W\sin\alpha$



When the body just starts to slide down:

$f = F$

$$\mu = \frac{F}{N} = \frac{w\sin\alpha}{W\cos\alpha} = \tan\alpha$$



We also have :

$\mu = \tan\phi$

hence,

$\varphi = \alpha$

Which means the maximum value of incline of a plane that allows a body to slide down under the effect of its weight is equal to the angle of friction ϕ

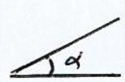
$$\alpha = \varphi = \tan^{-1}\mu$$

Example

What is the minimum angle of incline of a plane that allows a body to slide down under its weight if friction coefficient is 0.32.

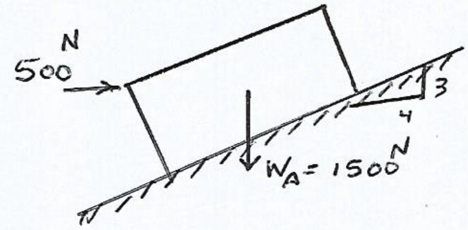
Solution

$$\alpha = \varphi = \tan^{-1}\mu$$
$$= \tan^{-1}(0.32) = 17.74^\circ$$



Example

Body A in the figure weighs 1500 N. Coefficient of friction between body A and the inclined plane is 0.4. Determine whether the body is in a static condition or moving.



Solution

$\sum Fx' = 0$

$500 * \frac{4}{5} - 1500 * \frac{3}{5} + f = 0$

$f = 500N$

$\sum Fy' = 0$

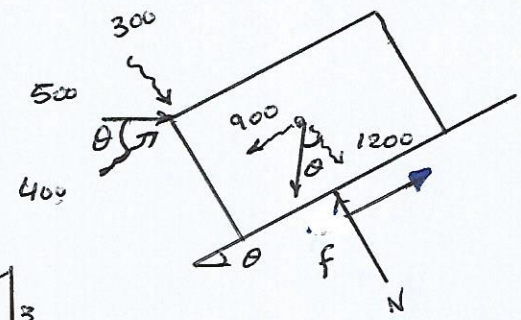
$-500 * \frac{3}{5} - 1500 * \frac{4}{5} + N = 0$

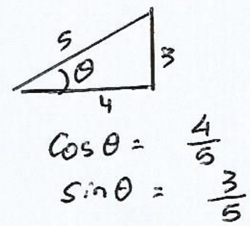
$N = 1500N$

$f = \mu N = 0.4 * 1500 = 600^{N}$

$f \langle F$

static condition

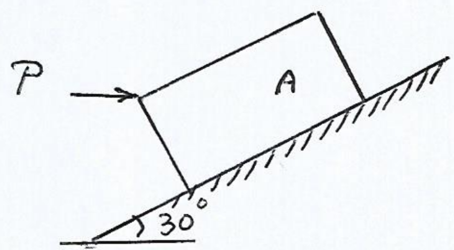




Example

Find the range of values of force P to keep body A in equilibrium state.

$W_A = 2000^N$, $\mu = 0.3$.



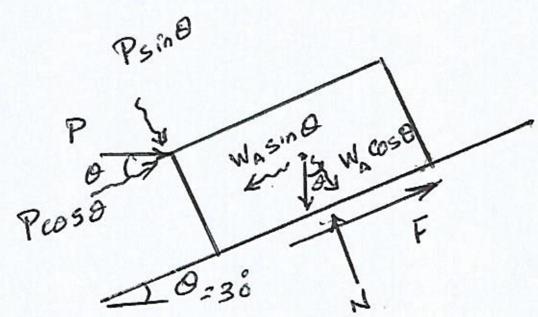
Solution

Check possibility of slipping downwards under the weight:

$\alpha = \tan^{-1} \mu = 16.7^{\circ} < 30^{\circ}$

$\therefore$ slipping down is possible

Body A slides down .. find P_{min}



$\sum Fx = 0$

$P\cos\theta - W_A \sin\theta + F = 0$

$0.866P - 0.5 * 2000 + 0.3N = 0$

$0.866P + 0.3N = 1000 \quad ..... (1)$

$\sum Fy = 0$

$-P\sin\theta - W_A \cos\theta + N = 0$

$-0.5P + N = 1732 \quad ..... (2)$

$(2) * 0.3 \quad -0.15P + 0.3N = 519.6$

$(1) - (2) * 0.3$

$1.016P = 480.4$

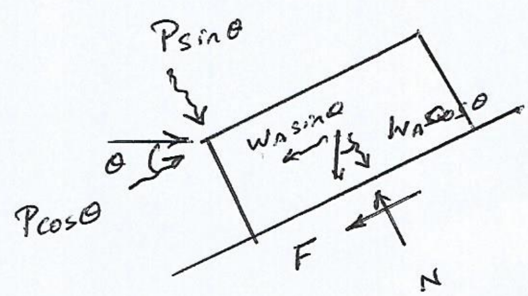
$P_{min} = 472.8^N$

Body A slides upwards ...... find P_{max}

$\sum Fy = 0$

$P\cos\theta - W_A \sin\theta - F = 0$

$0.866P - 0.3N = 1000 \quad ...... (1)$



$\sum Fx = 0$

$-P\sin\theta - W_A \cos\theta + N = 0$

$-0.5P + N = 1732 \quad ...... (2)$

$(2) * 0.3$

$-0.15P + 0.3N = 519.6$

$(1) + 0.3 * (2)$

$0.716P = 1519.6$

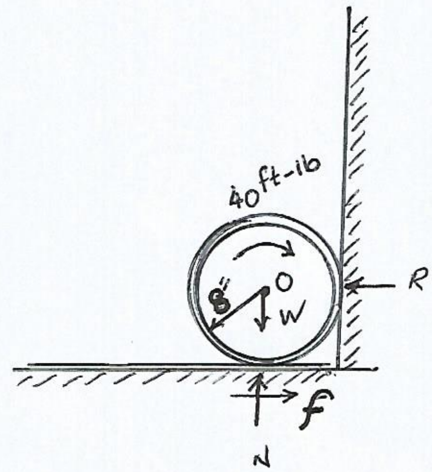
$P_{max} = 2122.3^N$

Range of P is : 472.8^N to 2122.3^N

Example

Coefficient of friction between the 200 Ib homogeneous cylinder in the figure and horizontal plane is 0.4. The vertical plane is smooth. Determine the frictional force exerted on the cylinder by the horizontal surface. Determine whether the cylinder is in static condition or not.

Solution



$\sum M_O = 0$

$40 - f * \frac{8}{12} = 0$

$f = 60 \ Ib$

$\sum Fy = 0$

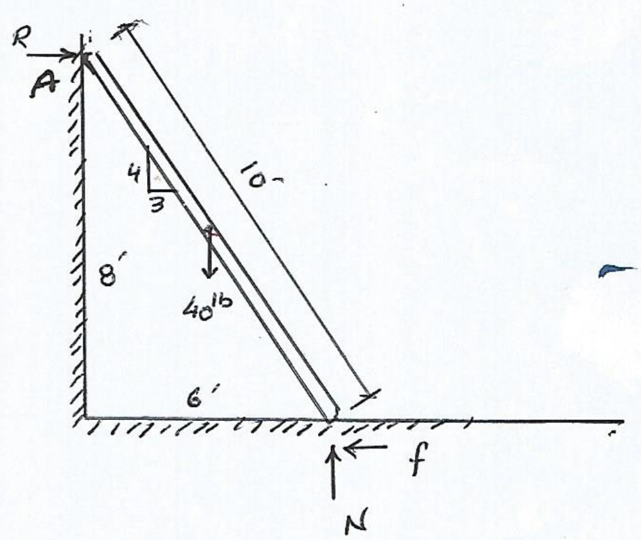
$N = 200 \ Ib$

$F = 200 * 0.4 = 80 \ Ib$

$f < F$ *static condition*

Example

The homogeneous 10ft bar shown in the figure weighs 40 Ib. The vertical wall is smooth and the horizontal plane has a coefficient of friction of 0.4. Determine the forces acting on the bar.



Solution

$\sum Fy = 0$

$N = 40\ Ib$

$\sum M_A = 0$

$40 * 3 - 40 * 6 + f * 8 = 0$

$f = 15\ Ib$

$\sum F_x = 0$

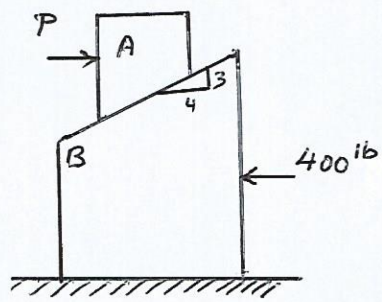
$R = 15\ Ib$

$F = 0.4 * 40 = 16\ Ib$

$f < F$ *No motion*

Example

In the figure shown, block A weighs 1000 Ib and block B weighs 2000 Ib. Coefficient of friction between A and B is 0.2 and between B and the horizontal surface is 0.1. Determine the maximum value of the force P for which body A will be in equilibrium.



Solution

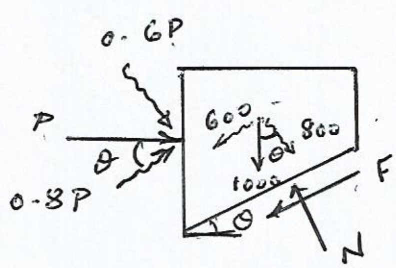
Check the possibility that body A slides upward over body B

$F = 0.2N$

$N = 5F$

$\theta = \tan^{-1}\frac{3}{4}$

$\sin\theta = 3/5, \cos\theta = 4/5$



$\sum Fx' = 0$

$0.8P - 600 - F = 0 \ldots (1)$

$\sum Fy' = 0$

$-0.6P - 800 + 5F = 0 \ldots. (2)$

$(1) * 5$

$4P - 3000 - 5F = 0$

$(2) + 5(1)$

$3.4P = 3800 \quad P = 1117.65\ Ib$

Body A and B together slide to the right on the surface

$\sum Fy = 0$

$N = 1000 + 2000 = 3000\ Ib$

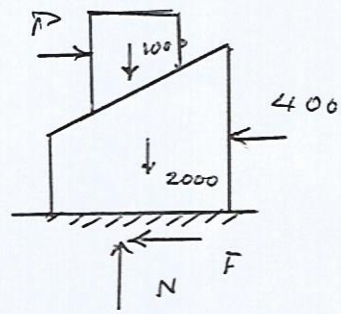
$F = 0.1 * 3000 = 300\ Ib \leftarrow$

$\sum Fx = 0$

$P - 400 - 300 = 0$

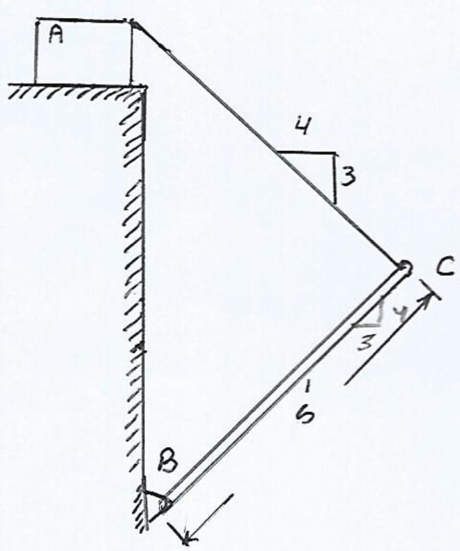
$P = 700\ Ib$

$\therefore P_{max} = 700\ Ib$



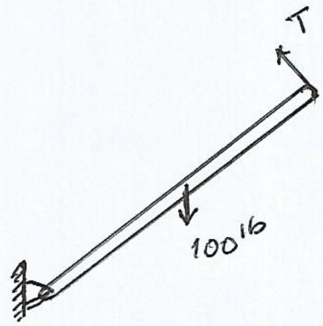
Example

Coefficient of friction between the 50 Ib body A and the plane is 0.5. The homogeneous bar BC weighs 100 Ib, Determine the forces acting on body A.



Solution

BC as F.B.D



$\sum M_B = 0$

$100*1.5 - T*5 = 0$

$T = 30\ Ib$

Body A as F.B.D

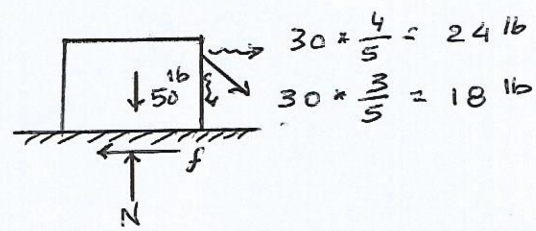
$\sum Fy = 0$

$N = 50 + 18 = 68\ Ib$

$\sum Fx = 0$

$f = 24\ Ib$

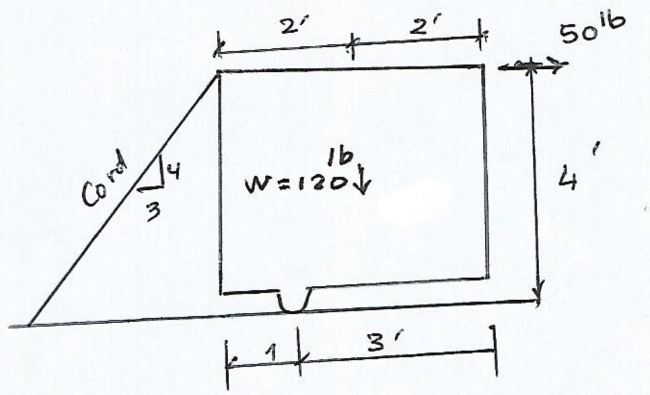
$Check\ F = 0.5*68 = 34\ Ib\ (Equilibrium)$



Example

The 120 Ib homogeneous body A is in equilibrium as shown in the figure. Determine the frictional force acting on the body and the minimum coefficient of friction required.

Solution



$\sum M_O = 0$

$120*1 + 50*4 - $ [illegible] $\frac{4}{5}$ [illegible] $= 0$

$0.6T*4 - 0.8T*1$

$T = 100\ Ib$

$\sum Fx = 0$

$50 - 100*\frac{3}{5} + f = 0$

$f = 10\ Ib$

$\sum Fy = 0$

$-120 - 100*\frac{4}{5} + N = 0$

$N = 200\ Ib$

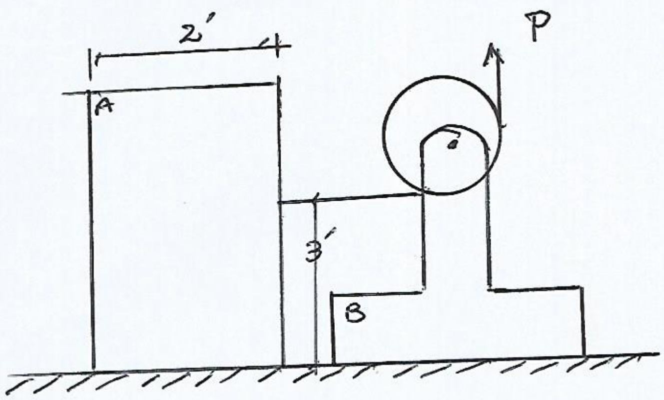
$For\ mnimum\ \mu \quad f = F$

$F = \mu N$

$10 = \mu_{\min} * 200 \qquad \mu_{\min} = 0.05$

Example

Each of block A and block B weighs 300 Ib. Coefficients of friction between A and B and the floor are 0.4 and 0.6 respectively. The pulley is frictionless. Determine the maximum value of P for which the system is in equilibrium. Block A is homogeneous.



Solution

Body A slides to the right:

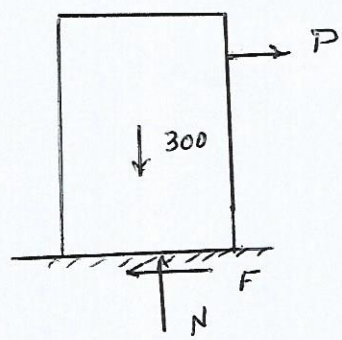
$\sum Fy = 0$

$N = 300\ Ib$

$F = \mu N = 0.4 * 300 = 120\ Ib$

$\sum Fx = 0$

$P = F = 120\ Ib$

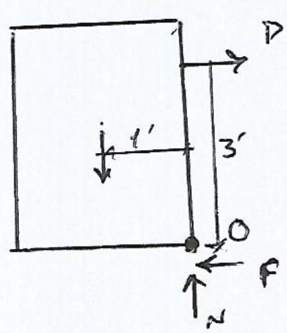


Body A overturns about O:

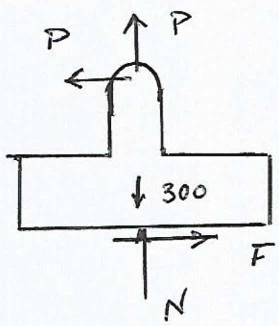
$\sum M_O = 0$

$P * 3 - 300 * 1 = 0$

$P = 100\ Ib$



Body B slides to the left :



$\sum Fy = 0$

$P + N = 300$

$\sum Fx = 0$

$P = F = 0.6N$

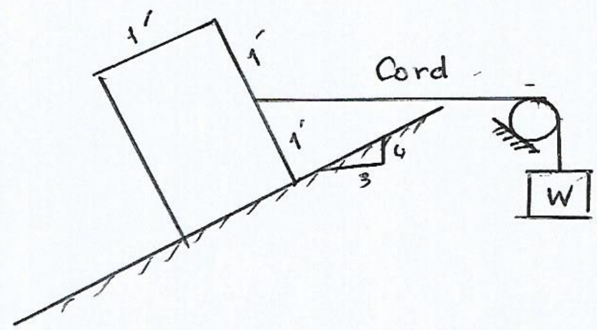
$N = 1.667P$

$P + 1.667P = 300 \quad 2.667P = 300 \quad \therefore P = 112.5\ Ib$

$P_{max} = 100\ Ib$

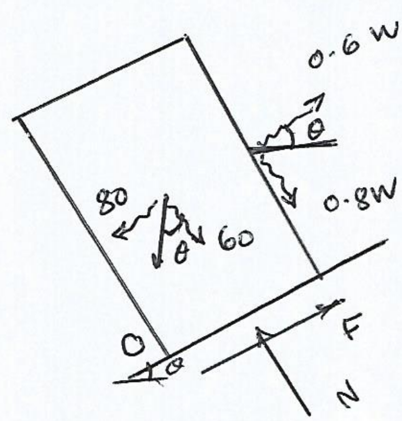
Example

A homogeneous block 1 ft square by 2 ft tall weighs 100 Ib. The block is supported by an inclined plane and the horizontal cord as indicated in the figure. Coefficient of friction between the block and the plane is 0.5. Determine the minimum value of W such that the block maintains equilibrium.



Solution

Block A slips downwards



$\sum Fx' = 0$

$0.6W + F - 80 = 0$

$0.6W + F = 80 \quad \text{..... (1)}$

$\sum Fy' = 0$

$-0.8W + N - 60 = 0$

$-0.8W + 2F = 60 \quad \text{...... (2)}$

$(1)*2$

$1.2W + 2F = 160$

$(1)*2-(2)$

$2W = 100 \qquad W = 50\ lb$

Block A tips over point O

$-80*1 + 60*0.5 + 0.6W*1 + 0.8W*1 = 0$

$-50 + 1.4W = 0 \qquad W = 35.7\ lb$

$\therefore W_{min} = 50\ lb$

CENTROID

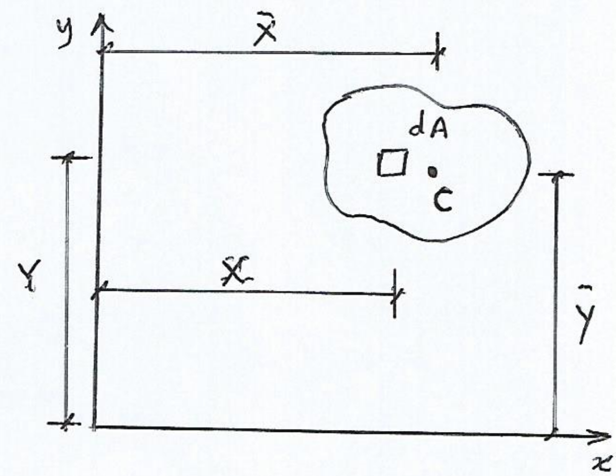
Centroid is the center of the area, where first moment of the area equals to zero.

If $\bar{x}$ *and* $\bar{y}$ are the coordinates of the Centroid, then moments of the area about axes passing through $\bar{x}$ *and* $\bar{y}$ are zero.

$$\bar{x} = \frac{\sum My}{A}$$

$$\bar{y} = \frac{\sum Mx}{A}$$

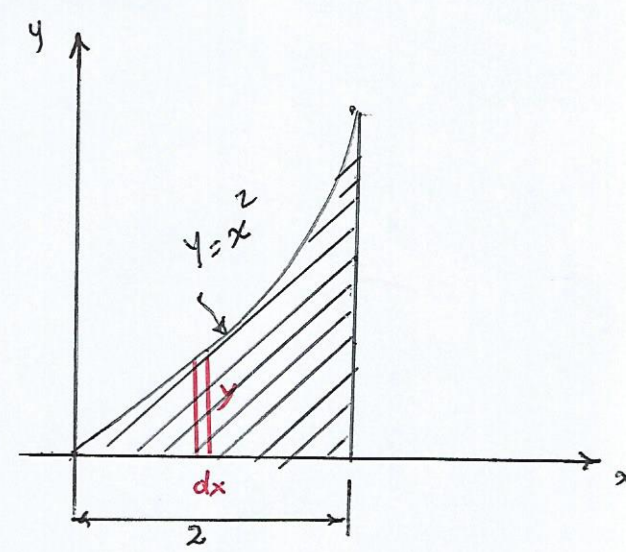
Where Mx and My are moments of the area with respect to axes x and y respectively (cm^3, in^3, ft^3,...)



$$\bar{X} = \frac{\int xdA}{\int dA} \quad , \quad \bar{y} = \frac{\int ydA}{\int dA}$$

Example

Locate the Centroid of the shaded area.



Solution

Method 1 – Vertical strips

$$A = \int ydx$$

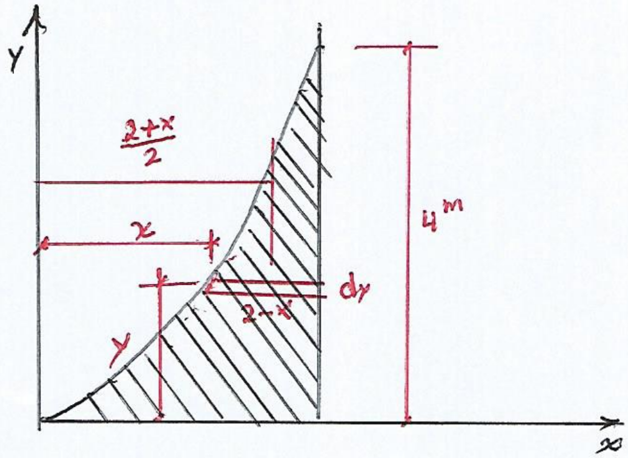
$$= \int x^2 dx = \frac{x^3}{3} = 2.667\ m^2$$

$$M_y = \int_0^2 ydx(x) = \int x^3 dx = \frac{x^4}{4} = 4.0\ m^3$$

$$M_x = \int_0^2 ydx(\frac{y}{2}) = \frac{1}{2}\int y^2 dx = \frac{1}{2}\int x^4 dx = \frac{x^5}{10} = 3.2\ m^3$$

$$\bar{x} = \frac{My}{A} = \frac{4}{2.667} = 1.5\ m$$

$$\bar{y} = \frac{Mx}{A} = \frac{3.2}{2.667} = 1.2\ m$$



Method 2 - Horizontal strips

$$A=\int_0^4 (2-x)dy$$

$$=\int_0^4 (2-y^{0.5})dy$$

$$=2y-\frac{2}{3}y^{1.5}=2.667\ m^2$$

$$M_y=\int_0^4 (2-x)dy\,\frac{2+x}{2}=\frac{1}{2}\int(4-x^2)\,dy=\frac{1}{2}\int(4-y)\,dy=4.0\ m^3$$

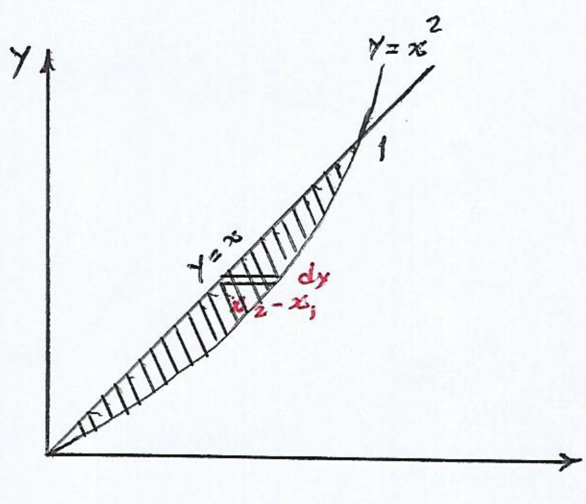
$$M_x=\int_0^4 (2-x)dy(y)=\int y(2-y^{0.5})\,dy=\int(2y-y^{1.5})dy$$

$$=y^2-\frac{y^{2.5}}{2.5}=16-32/2.5=3.2\ m^3$$

$\overline{X}=1.5m,\quad \overline{y}=1.2m$

Example

Locate the Centroid of the shaded area.



Solution

y1=y2

$x=x^2 \quad x(x-1)=0 \qquad x=0,\ x=1$

$x_2-x_1=y^{0.5}-y$

$$A = \int_0^1 (y^{0.5} - y)dy$$

$$= \left[\frac{2}{3}y^{1.5} - \frac{1}{2}y^2\ \right] = \frac{2}{3} - \frac{1}{2} = 0.1667\ m^2$$

$$Mx = \int_0^1 (y^{0.5} - y)dy(y) = \int (y^{1.5} - y^2)dy$$

$$= \frac{2}{5}y^{2.5} - \frac{1}{3}y^3] = \frac{2}{5} - \frac{1}{3} = 0.0667\ m^2$$

$$My = \int_0^1 (y^{0.5} - y)\frac{1}{2}(y^{0.5} + y)dy$$

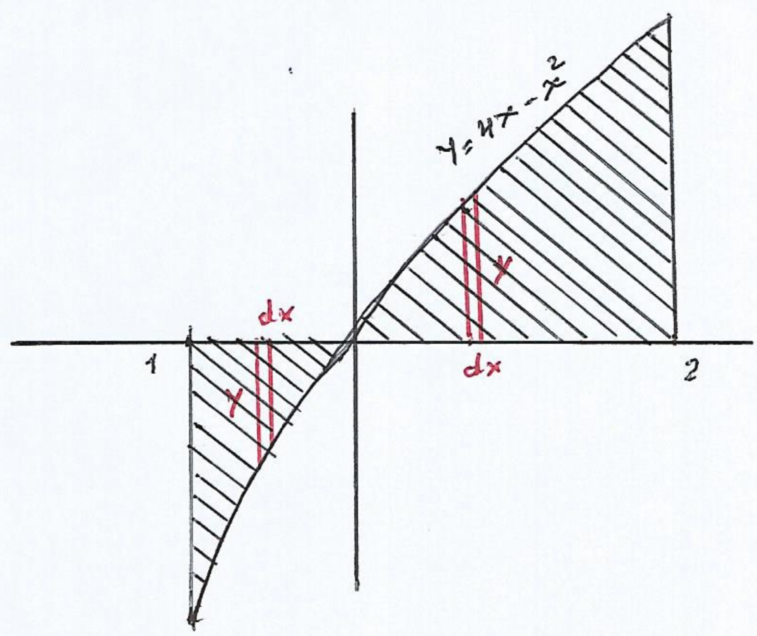
$$= \int (y - y^2)dy \ = \frac{y^2}{2} - \frac{y^3}{3}] = 0.0833\ m^2$$

$$\bar{x} = \frac{0.08333}{0.1667} = 0.5\ m$$

$$\bar{y} = \frac{0.0667}{0.1667} = 0.4\ m$$

Example

Find the coordinates for the Centroid of the shaded area shown in the figure



Solution

$A = A_1 + A_2$

$$A_1 = \left| \int_{-1}^{0} y dx \right|$$

$$= \left| \int_{-1}^{0} (4x - x^2) dx \right|$$

$$= \left| [2x^2 - \frac{1}{3}x^3] \right| = \left| [0 - 2 + \frac{1}{3}(-1)] \right| = 2.333$$

$$A_2 \left| \int_{0}^{2} y dx \right|$$

$$= \left| \int_{0}^{2} (4x - x^2) dx \right|$$

$$= \left| [2x^2 - \frac{1}{3}x^3] \right| = \left| 8 - \frac{8}{3} \right| = 5.333$$

$$A = 2.333 + 5.333 = 7.666\ in^2$$

$$M_y = -M_{y1} + M_{y2}$$

$$M_{y1} = \left| \int_{-1}^{0} y dx(x) \right|$$

$$= \left| \int_{-1}^{0} (4x^2 - x^3) dx \right|$$

$$= \left| [\frac{4}{3}x^3 - \frac{1}{4}x^4] \right| = \left| [0 - \frac{4}{3}(-1) + \frac{1}{4}(1)] \right| = 1.583$$

$$M_{y2} = \left| \int_{0}^{2} y dx(x) \right|$$

$$= \left| \int_{0}^{2} (4x^2 - x^3) dx \right|$$

$$= \left| [\frac{4}{3}x^3 - \frac{1}{4}x^4] \right| = [\frac{32}{3} - 4] = 6.667$$

$$M_y = -1.583 + 6.667 = 5.083\ in^3$$

$$M_x = -M_{x1} + M_{x2}$$

$$M_{x1} = \left| \int_{-1}^{0} y dx (\frac{y}{2}) \right|$$

$$= \left| \frac{1}{2} \int_{-1}^{0} (4x - x^2)^2 dx \right|$$

$$= \left| \frac{1}{2} \int_{-1}^{0} (16x^2 - 8x^3 + x^4)\, dx \right|$$

$$= \left| \frac{1}{2} [\frac{16}{3} x^3 - 2x^4 + \frac{x^5}{5}] \right| = \left| \; = \frac{1}{2}[0 - \frac{16}{3}(-1) + 2(1) - \frac{1}{5}(-1)] \right| = 7.533$$

$$M_{x2} = \left| \frac{1}{2} [\frac{16}{3} x^3 - 2x^4 + \frac{x^5}{5}] \right| = [\frac{16}{3}(8) - 2(16) + \frac{32}{5}] = 8.533$$

$$Mx = -7.533 + 8.533 = 1 \; in^3$$

$$\bar{x} = \frac{5.083}{7.666} = 0.663 \; in$$

$$\bar{y} = \frac{1}{7.666} = 0.13 \; in$$

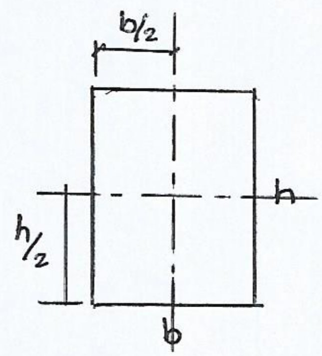
Centroid Coordinates for some Regular Geometric Shapes

Rectangular

A=bh

$$\bar{x} = \frac{b}{2}$$

$$\bar{y} = \frac{h}{2}$$

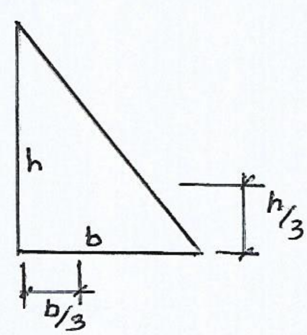


Triangular

$$A = \frac{bh}{2}$$

$$\bar{x} = \frac{b}{3}$$

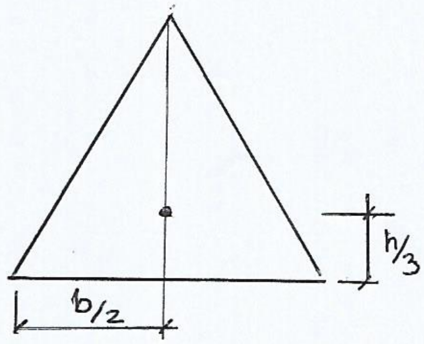
$$\bar{y} = \frac{h}{3}$$



$$A = \frac{bh}{2}$$

$$\bar{x} = \frac{b}{2}$$

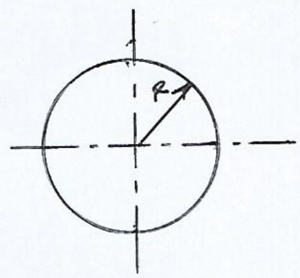
$$\bar{y} = \frac{h}{3}$$



Circular

$$A = \pi R^2$$

$$\bar{x} = \bar{y} = 0$$

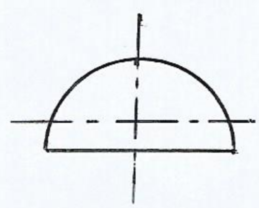


Half a circle

$$A = \pi R^2 / 2$$

$$\bar{x} = 0$$

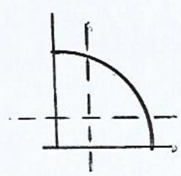
$$\bar{y} = \frac{4R}{3\pi}$$



Quarter a circle

$$A = \pi R^2 / 4$$

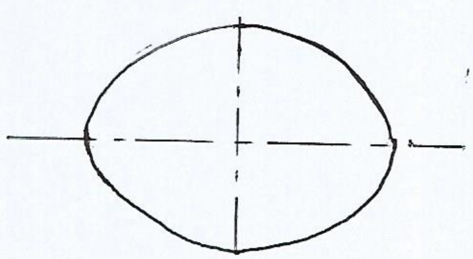
$$\bar{x} = \bar{y} = \frac{4R}{3\pi}$$



Ellipse

$$A = \pi ab$$

$$\bar{x} = \bar{y} = 0$$

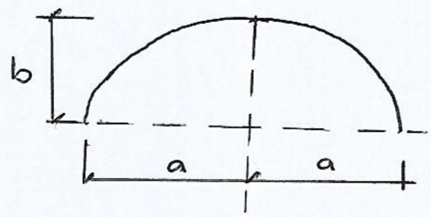


Half an Ellipse

$$A = \pi ab / 2$$

$$\bar{x} = 0$$

$$\bar{y} = \frac{4b}{3\pi}$$

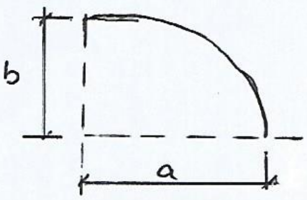


Quarter an Ellipse

$$A = \pi ab / 4$$

$$\bar{x} = \frac{4a}{3\pi}$$

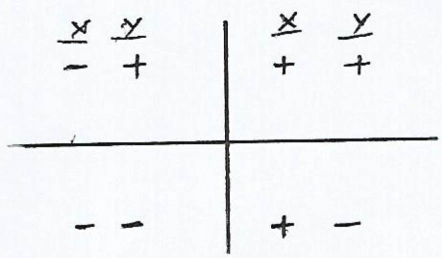
$$\bar{y} = \frac{4b}{3\pi}$$



Compound Areas

If the area is composed of a number of regular areas, we deal with the summation of these areas and their moments separately with respect to axes x and y.

- Holes are regarded as negative areas.
- The signs for distances from the Centroid of each sub area to x and y axes are taken according to the sign convention for the four quarters as shown in the figure.



$$A = \sum Ai = A1 + A2 + ...$$

$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i}$$

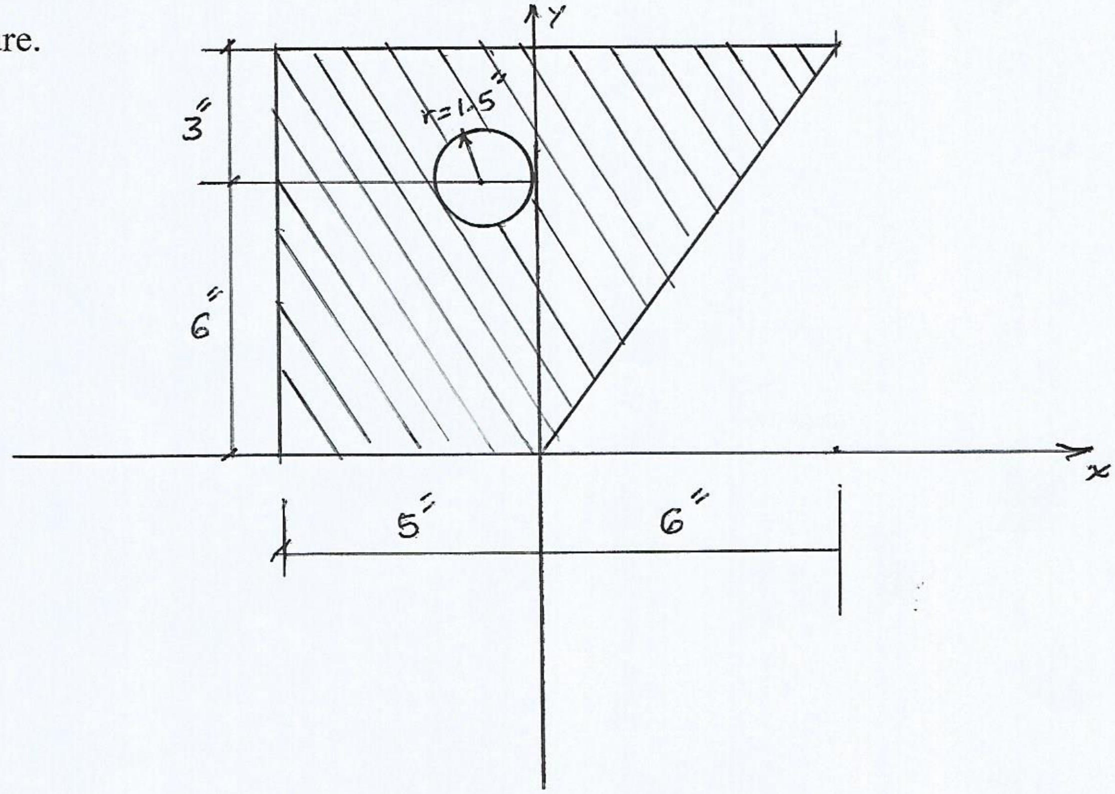
$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i}$$

x_i : ~~vertical~~ Horizontal distance from the Centroid of sub area to y axis.

y_i : vertical distance from the Centroid of sub area to x axis.

Example

Determine the coordinates of the Centroid of the shaded area shown in the figure.



Solution

A1=5*9= 45 in2

A2=6*9/2= 27 in2

$$A3 = \pi(1.5)^2 = 7.07\ in^2$$

$$A = \sum A = 45 + 27 - 7.07$$

$$= 64.93\ in^2$$

$$\sum M_x = 45 * 4.5 + 27 * 6 - 7.07 * 6 = 322.08\ in^3$$

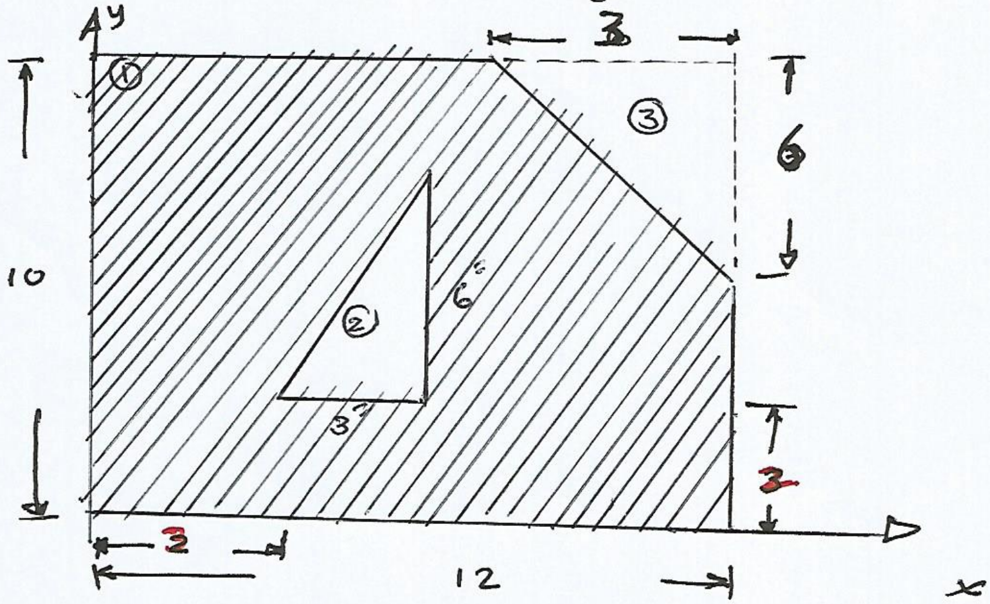
$$\sum M_y = -45 * 2.5 + 27 * 2 - 7.07 * (-1.5) = -47.898$$

$$\bar{x} = \frac{-47.898}{64.93} = -0.73\ in$$

$$\bar{y} = \frac{322.08}{64.93} = 4.96\ in$$

Example

Locate the Centroid of the shaded area shown in the figure.



solution

$$A_1 = 12*10 = 120\ in^2$$

$$A_2 = \frac{-3*6}{2} = -9\ in^2$$

$$A_3 = \frac{-3*6}{2} = -9\ in^2$$

$$A = 120 - 9 - 9 = 102\ in^2$$

$$\sum M_x = 120*5 - 9*4 - 9*8 = 492\ in^3$$

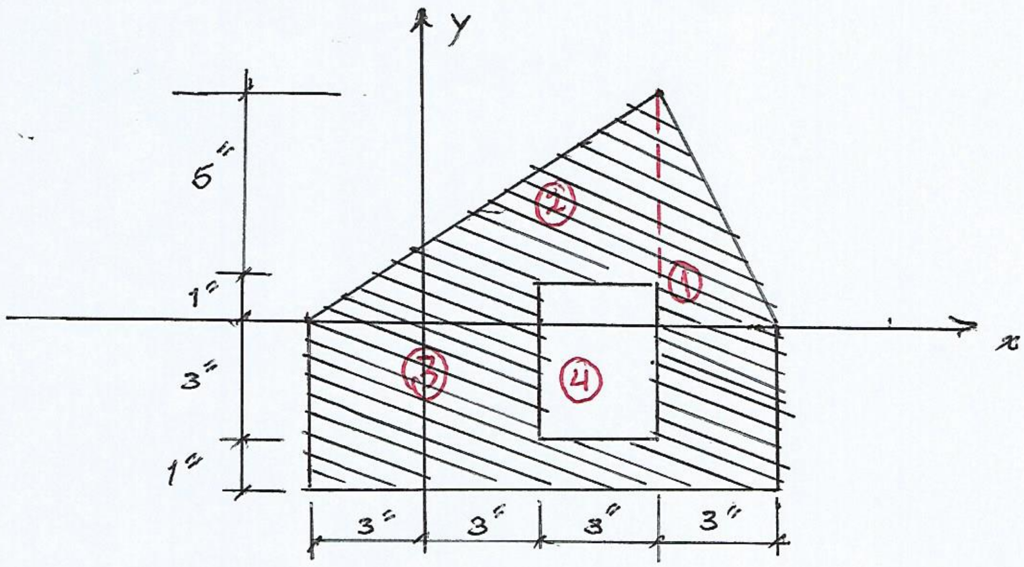
$$\sum M_y = 120*6 - 9*5 - 9*11 = 576\ in^3$$

$$\bar{x} = \frac{576}{102} = 5.66\ in$$

$$\bar{y} = \frac{492}{102} = 4.82\ in$$

Example

Determine the coordinates x and y for the Centroid of the shaded area shown in the figure.



Solution

$$A_1 = \frac{3*6}{2} = 9\ in^2$$

$$A_2 = \frac{9*6}{2} = 27\ in^2$$

$$A_3 = 4*12 = 48 in^2$$

$$A_4 = -3*4 = -12\ in^2$$

$$A = 9 + 27 + 48 - 12 = 72\ in^2$$

$$\sum M_x = 9*2 + 27*2 + 48(-2) - 12(-1) = -12\ in^3$$

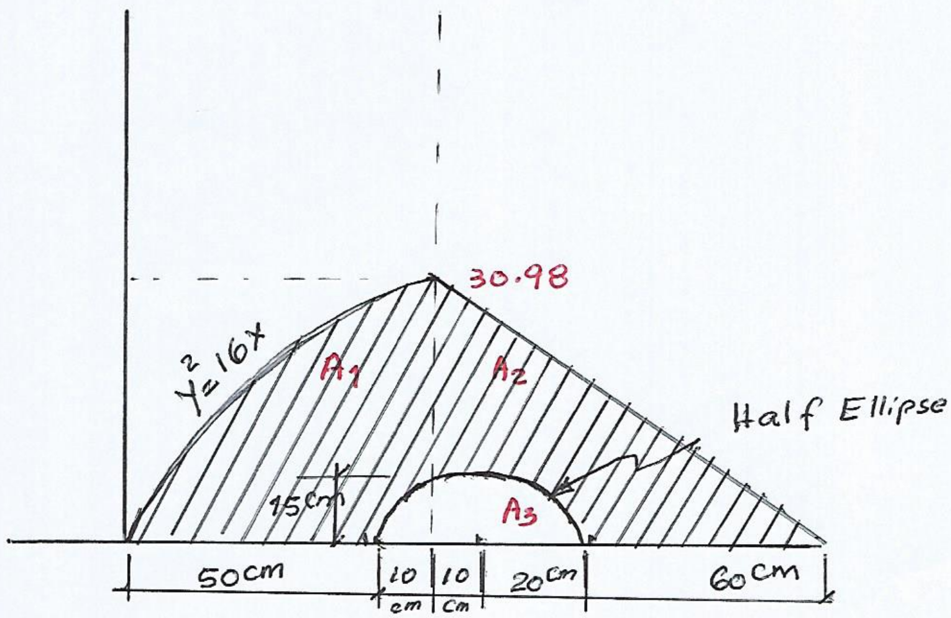
$$\sum M_y = 9*7 + 27*3 + 48*3 - 12*4.5 = 234\ in^3$$

$$\bar{x} = \frac{234}{72} = 3.24\ in$$

$$\bar{y} = \frac{-12}{72} = -\ 0.167\ in$$

Example

Determine the moment of the shaded area with respect to the axis a-a



solution

$$M_1 = \int_0^{60} y dx(60 - x)$$

$$= 4\int x^{0.5}(60 - x)dx$$

$$= [\frac{240 * 2}{3}x^{1.5} - \frac{4 * 2}{5}x^{2.5}]$$

$$= -29744.5\ cm^3$$

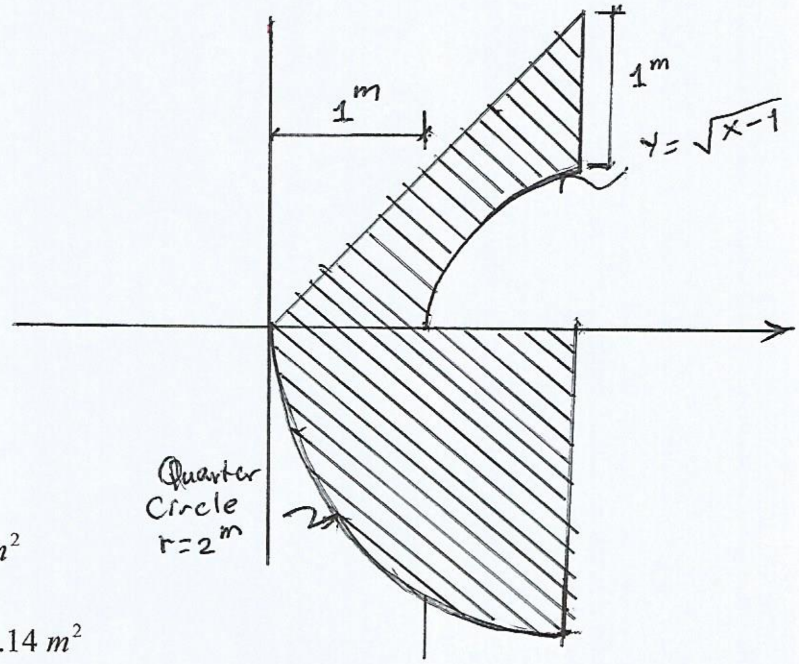
$$M_2 = \frac{90 * 30.98}{2} * 30 = 41823\ cm^3$$

$$M_3 = \frac{\pi(20)(15)(10)}{2} = -4712.4\ cm^3$$

$$M = 41823 - 29744.5 - 4712.4 = 7366.1\ cm^3$$

Example

Determine y coordinate for the Centroid of the shaded area shown



solution

$$A_1 = \frac{2*2}{2} = 2\ m^2$$

$$A_2 = \frac{\pi(2)^2}{4} = 3.14\ m^2$$

$$A_3 = \int_1^2 ydx = \int (x-1)^{0.5} dx = \frac{2}{3}(x-1)^{1.5} = 0.667\ m^2$$

$$A = 2 + 3.14 - 0.67 = 4.47\ m^2$$

$$A = 63.27\ m^2$$

$$M_1 = 2 * \frac{2}{3} = 1.333\ m^3$$

$$M_2 = 3.14(\frac{-4*2}{3\pi}) = -2.67\ m^3$$

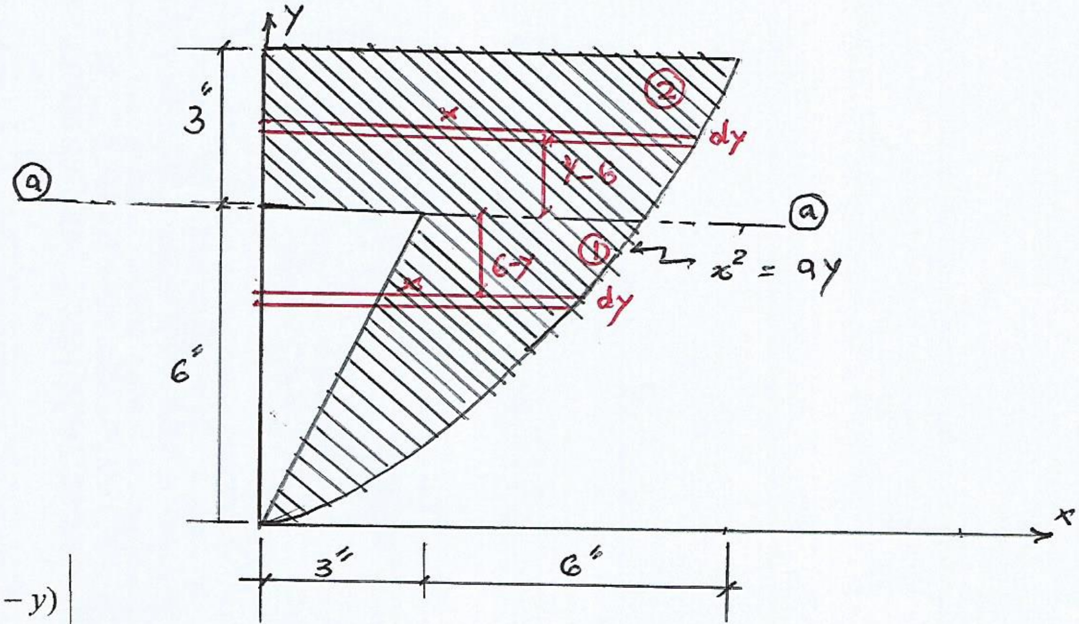
$$M_3 = -\left| \int ydx(\frac{y}{2}) \right| = -\frac{1}{2}\left| \int (x-1)dx \right| = -\frac{1}{2}\left| \frac{(x-1)^2}{2} \right| = -0.25\ m^3$$

$$M_x = 1.33 - 2.67 - 0.25 = -1.59\ m^3$$

$$\bar{y} = -0.36\ m$$

Example

Determine the moment of the shaded area about axis a-a



Solution

$$M_1 = \left| \int_0^6 x dy (6-y) \right|$$

$$= \left| \int 3y^{0.5}(6-y)dy \right|$$

$$= \left| \int (18y^{0.5} - 3y^{1.5})dy \right|$$

$$= \left| 12y^{1.5} - 1.2y^{2.5} \right| = 70.54 \ in^3$$

$$M_2 = \left| \int_6^9 x dy (y-6) \right|$$

$$= \left| \int 3y^{0.5}(y-6)dy \right|$$

$$= \left| \int (3y^{1.5} - 18y^{0.5})dy \right|$$

$$= \left| 1.2y^{2.5} - 12y^{1.5} \right| = 38.14 \ in^3$$

$$M = -70.54 + 38.14 + \left(\frac{-3*6}{2}\right)(-2)$$

$$= -14.4 \ in^3$$

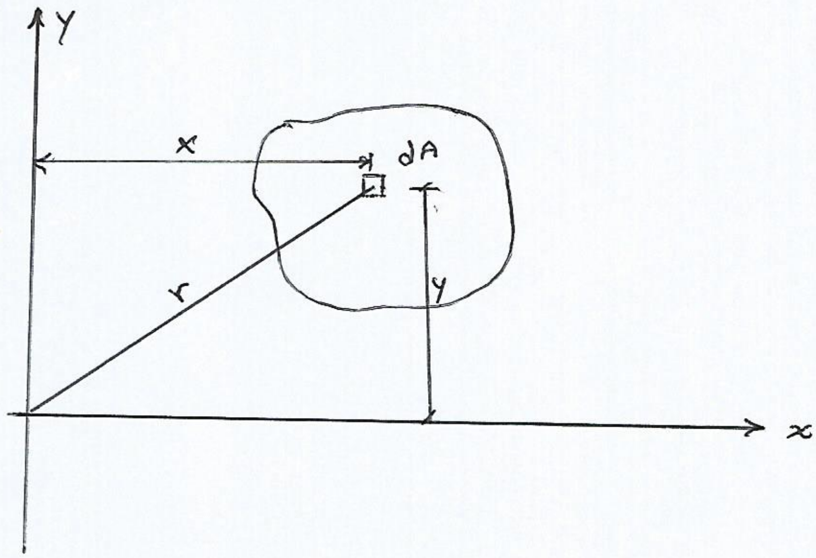
MOMENT OF INERTIA

Also called - Second moment of Area

$$I_x = \int y^2 dA$$

$$I_y = \int x^2 dA$$

$$J_O = \int r^2 dA$$
$$= \int (x^2 + y^2) dA$$
$$= \int x^2 dA + \int y^2 dA$$
$$= I_x + I_y$$



Where:

I_x: moment of Inertia (second moment of the area) with respect to x- axis

I_y: moment of Inertia (second moment of the area) with respect to y- axis

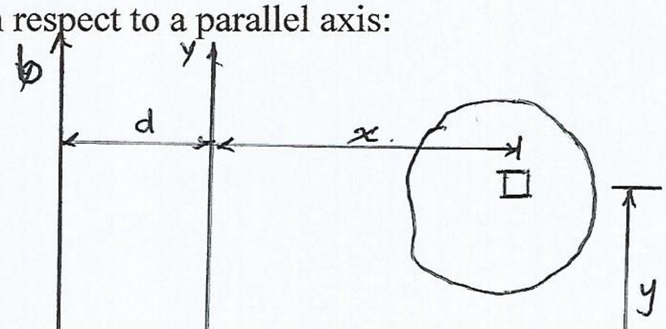
J_O: Polar moment of Inertia

Units for I_x, I_y and J_O are (length)^4 units like cm^4, in^4, ft4, m^4, …

I_x, I_y and J_O are all positive quantities.

Parallel axes theorem

To find the moment of inertia with respect to a parallel axis:



$$
\begin{aligned}
I_b &= \int (x+d)^2 dA \\
&= \int (x^2 + 2xd + d^2)\ dA \\
&= \int x^2 dA + 2d \int x dA + d^2 \int dA \\
&= I_x + 2dM_y + d^2 A
\end{aligned}
$$

When y passes through the Centroid of the area:

$M_y = 0 \quad and \quad 2dM_y = 0$

$\therefore I_b = I_x + Ad^2$

Similarly, we can write:

$I_b = I_y + Ad^2$

$J_b = J_b + Ad^2$

The parallel axis theorem states that the moment of inertia of an area is equal to the moment of inertia with respect to a parallel axis through the Centroid of the area plus the product of the area and the square of the distancebetween the two axes.

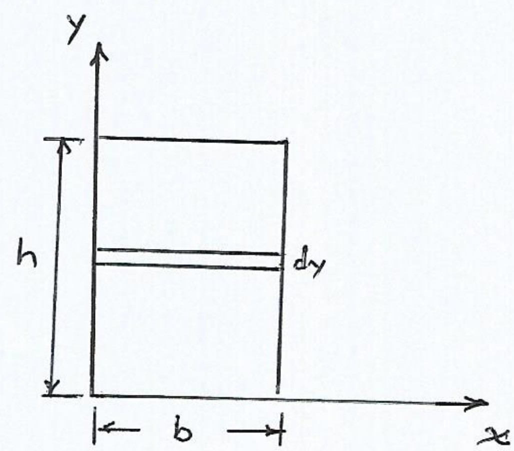
Notes:

- Moment of inertia of any area with respect to an axis through the Centroid is less than that of any parallel axis.
- Parallel axis theorem can only transform the moment of inertia from axis through the Centroid of the area.

Example

(a)Determine I_x and I_y for the rectangular area as shown in the figure.

(b) Determine I_x and I_y of the same area with respect to axes through the Centroid .



Solution

(1)

$$I_x = \int_0^h y^2 dA$$

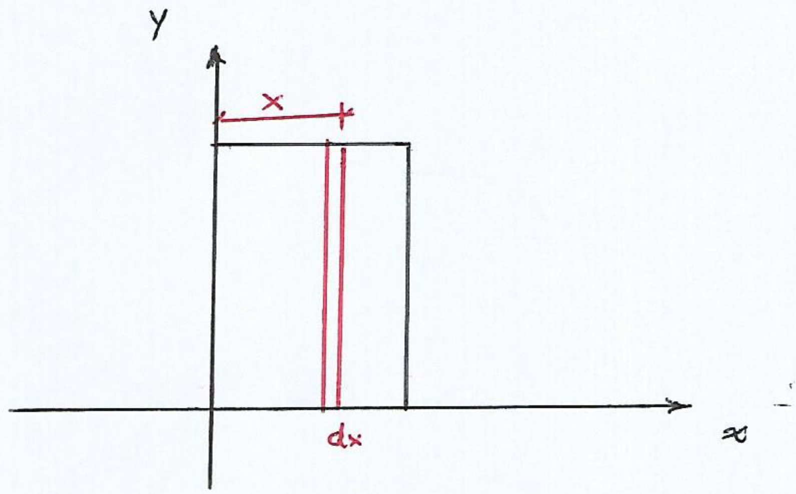
$$= \int_0^h y^2 (bdy)$$

$$= [\frac{by^3}{3}] = \frac{bh^3}{3}$$

$$I_y = \int_0^b x^2 dA$$

$$= \int_0^h x^2 (hdx)$$

$$= [\frac{hx^3}{3}] = \frac{hb^3}{3}$$



(2)

$$I_x = I_{xc} + Ad^2$$

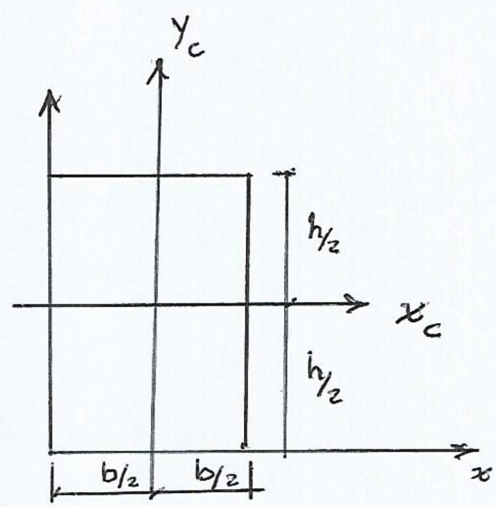
$$\frac{bh^3}{3} = I_{xc} + bh(\frac{h}{2})^2$$

$$I_{xc} = \frac{bh^3}{3} - \frac{bh^3}{4} = \frac{bh^3}{12}$$

$$I_y = I_{yc} + Ad^2$$

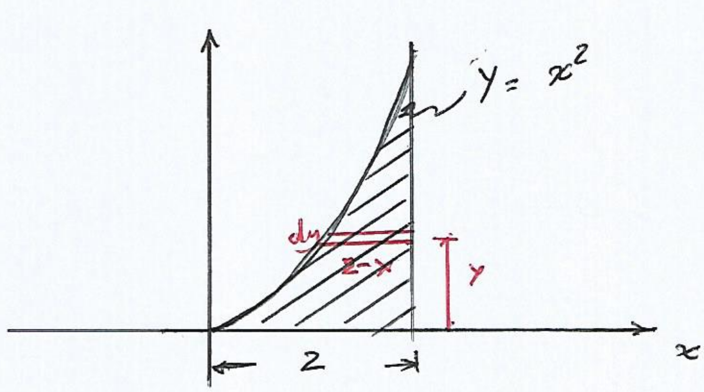
$$\frac{hb^3}{3} = I_{yc} + bh(\frac{b}{2})^2$$

$$I_{xc} = \frac{bh^3}{3} - \frac{hb^3}{4} = \frac{hb^3}{12}$$



Example

Determine I_x and I_y for the shaded area as shown in the figure.



Solution

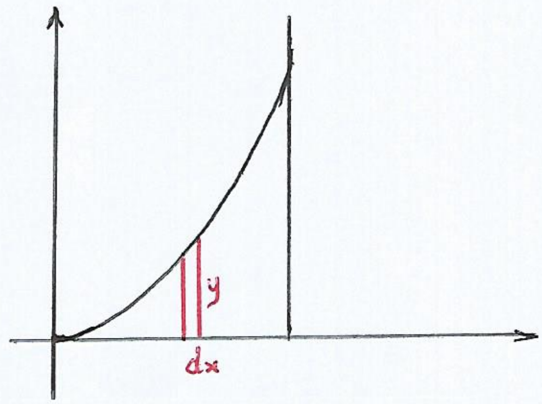
Method 1: using parallel strips- definition to find I

$$I_x = \int_0^4 y^2(2-x)dy$$

$$= \int_0^4 y^2(2-y^{0.5})dy$$

$$= \int (2y^2 - y^{2.5})dy$$

$$= [\frac{2y^3}{3} - \frac{y^{3.5}}{3.5}] = 6.095\ in^4$$



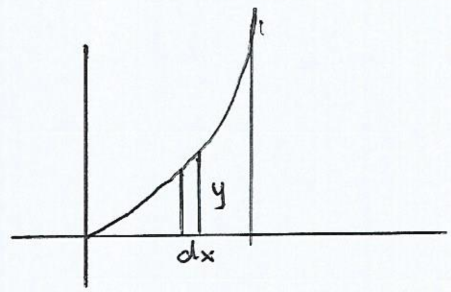
$$I_y = \int_0^2 x^2 y dx$$

$$= \int x^4 dx$$

$$= [\frac{x^5}{5}] = 6.4\ in^4$$

Method 2: using perpendicular strips- use I for summation of rectangular

Strips.

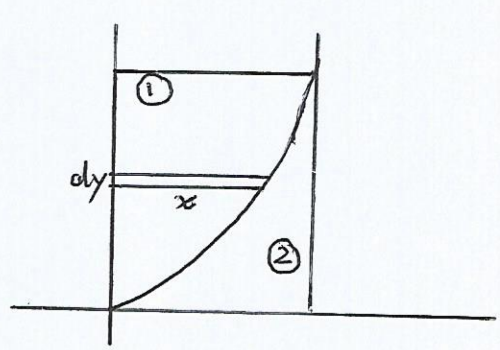


$$I_x = \int_0^2 \frac{y^3 dx}{3}$$

$$= \frac{1}{3}\int_0^2 x^6 dx = [\frac{x^7}{21}] = 6.095 \quad in^4$$

$$I_y = \frac{4(2)^3}{3} - \int_0^4 \frac{x^3 dy}{3}$$

$$= 10.667 - [\frac{y^{2.5}}{7.5}] = 6.4\ in^4$$



Moment of Inertia for some Regular shapes

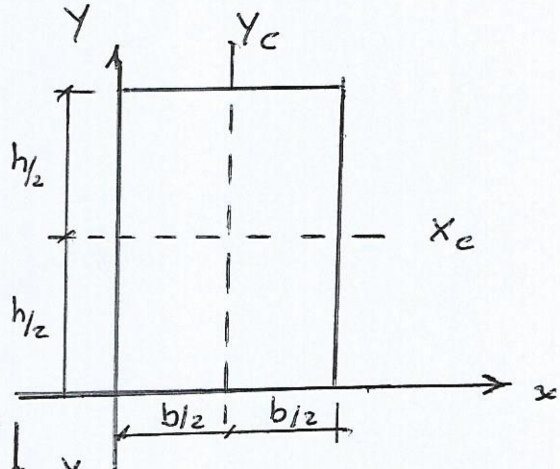
Rectangular

$$I_x = \frac{bh^3}{3}$$

$$I_y = \frac{hb^3}{3}$$

$$I_{xc} = \frac{bh^3}{12}$$

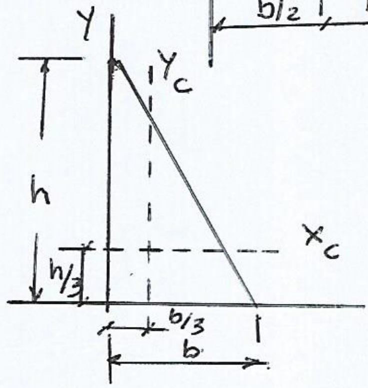
$$I_{yc} = \frac{hb^3}{12}$$



Triangular

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12}$$

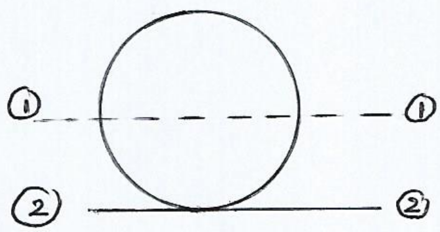
$$I_{xc} = \frac{bh^3}{36} \qquad I_{yc} = \frac{hb^3}{36}$$



Circle

$$I_{1\text{-}1} = \frac{\pi R^4}{4}$$

$$I_{2\text{-}2} = \frac{5\pi R^4}{4}$$

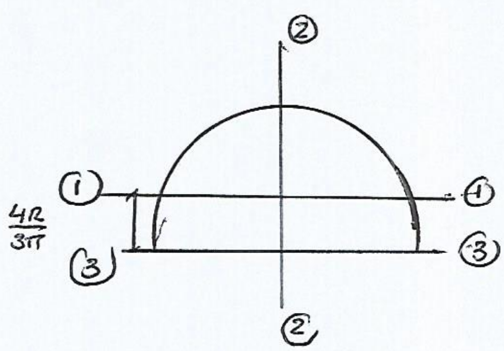


Half a circle

$$I_{1\text{-}1} = 0.11R^4$$

$$I_{2\text{-}2} = \frac{\pi R^4}{4}$$

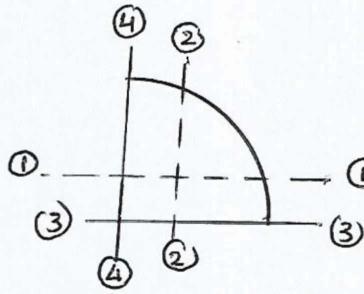
$$I_{3\text{-}3} = \frac{5\pi R^4}{4}$$



Quarter a circle

$$I_{1-1} = I_{2-2} = 0.055R^4$$

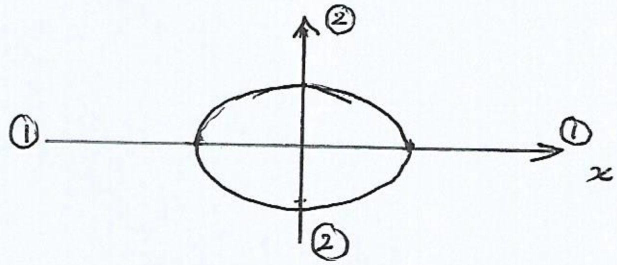
$$I_{3-3} = I_{4-4} = \frac{\pi R^4}{16}$$



Ellipse

$$I_{1-1} = \frac{\pi a b^3}{4}$$

$$I_{2-2} = \frac{\pi b a^3}{4}$$



Radius of Gyration

Radius of gyration is the length which, when squared and multiplied by the area, will give the moment of inertia of the area with respect to the given axis.

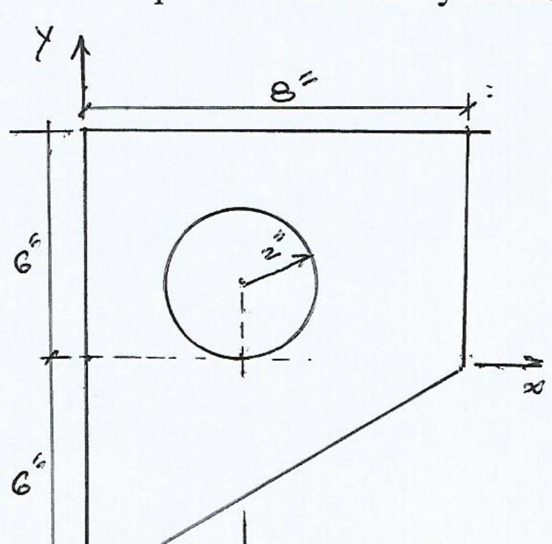
$$k_x = \sqrt{\frac{I_x}{A}}$$

$$k_y = \sqrt{\frac{I_y}{A}}$$

$$k_O = \sqrt{\frac{J_O}{A}}$$

Example

Determine the moments of inertia and radii of gyration of the shaded area with respect to axes x and y as shown in the figure.



solution

$$I_x = \frac{8(6)^3}{3} + \frac{8(6)^3}{12} - \frac{5\pi(2)^4}{4} = 576 + 144 - 62.83 = 657.17\ in^4$$

$$A = 8*6 + \frac{8*6}{2} - \pi(2)^2 = 59.43\ in^2$$

$$k_x = \sqrt{\frac{657.17}{59.43}} = 3.325\ in$$

$$I_x = \frac{6(8)^3}{3} + \frac{6(8)^3}{12} - (\frac{\pi(2)^4}{4} + \pi(2)^2(4)^2) = 1365.3 + 341.3 - 213.62 = 1492.98\ in^4$$

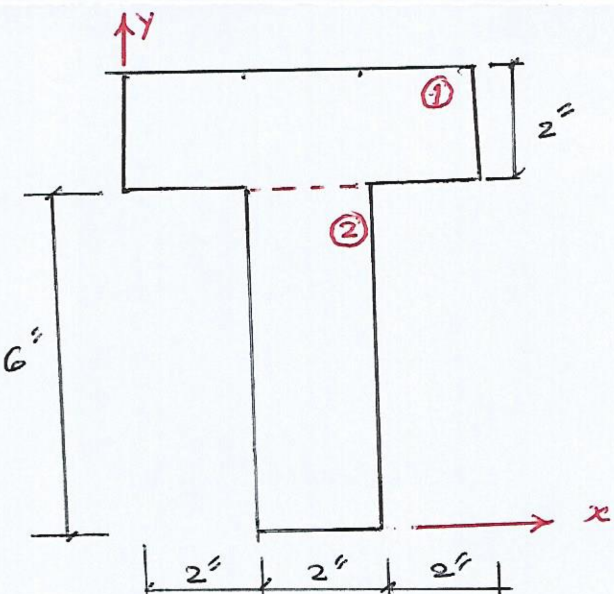
$$k_x = \sqrt{\frac{1492.98}{59.43}} = 5.01\ in$$

Example

Determine the moments of inertia of the T-section shown with respect to horizontal and vertical axes through its Centroid.

Solution

Location of the Centroid



$$A1 = 6*2 = 12\ in^2$$

$$A_2 = 6*2 = 12\ in^2$$

$$A = 24\ in^2$$

$$M_x = 12*7 + 12*3 = 120\ in^3$$

$$M_y = 12*3 + 12*3 = 72\ in^3$$

$$\bar{x} = \frac{72}{24} = 3\ in \quad \bar{y} = \frac{120}{24} = 5\ in$$

$$I_{xc1} = \frac{6(2)^3}{12} + 12(2)^2 = 4 + 48 = 52\ in^4$$

$$I_{xc1} = \frac{2(6)^3}{12} + 12(2)^2 = 36 + 48 = 84\ in^4$$

$$I_{xc} = 52 + 84 = 136\ in^4$$

$$I_{yc} = \frac{2(6)^3}{12} + \frac{6(2)^3}{12} = 36 + 4 = 40\ in^4$$

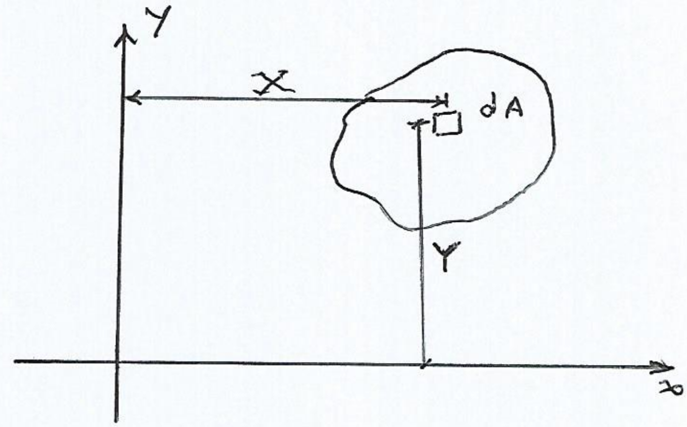
PRODUCT OF INERTIA

$I_{xy} = \int XY\,dA$

where

I_{xy} : *product of Inertia with respect to the origin*

I_{xy} *can be positive or negative*



Parallel Axis theorem

$I_{xy} = I_{xy\,C} + A\,\overline{xy}$

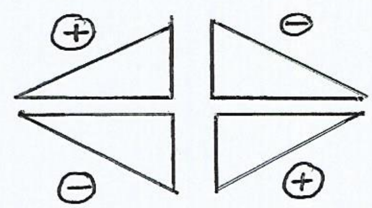
Note

I_{xy} is zero for every symmetrical shape like rectangular, circle, half circle… provided there is at least one axis of symmetry.

For a right angle triangle:

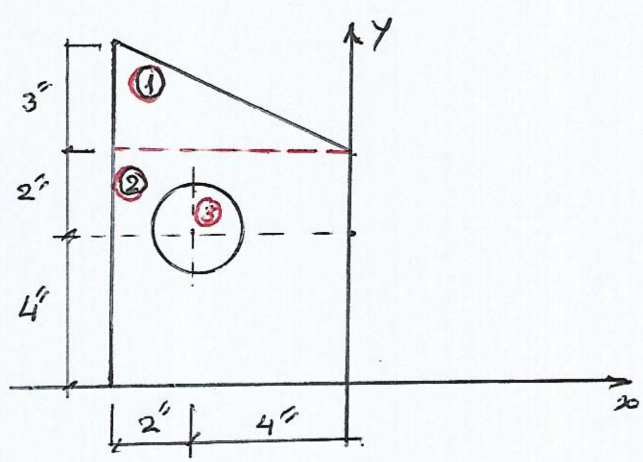
$I_{xy} = +/- \frac{b^2 h^2}{72}$

Take the sign according to the following sign convention depending on the direction of the right angle.



Example

Determine product of inertia I_{xy} for the shaded area shown with respect to axes through the origin.



Solution

$$I_{xy1} = \frac{-(6)^2(3)^2}{72} + \frac{6*3}{2}(-4)(7) = -256.5\ in^4$$

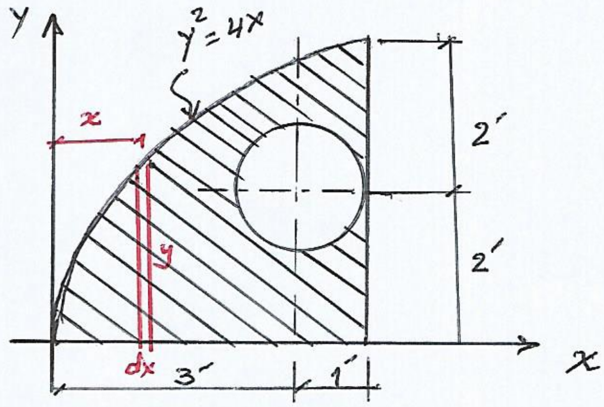
$$I_{xy2} = 0 + 6*6(-3)(3) = -324\ in^4$$

$$I_{xy3} = 0 + \pi(1)^2(-4)(-4) = -50.2655\ in^4$$

$$Ixy = -256.5 - 324 + 50.3655 = -530.235\ in^4$$

Example

Determine the product of inertia I_{xy} for the shaded area shown with respect to axes through the origin.



Solution

$$I_{xy1} = \int_0^4 x\frac{y}{2}ydx$$

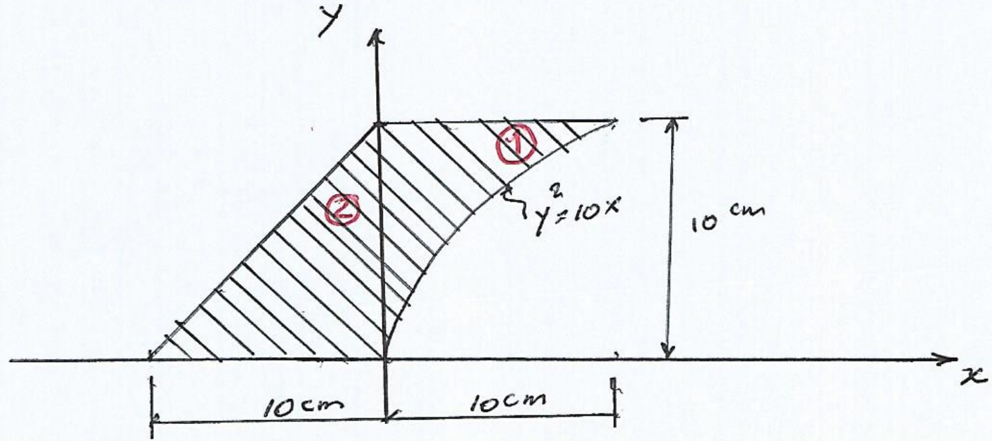
$$= \frac{1}{2}\int_0^4 4x^2dx$$

$$= \frac{1}{2}[\frac{4x^3}{3}] = 42.667\ ft^4$$

$$I_{xy2} = 0 + \pi(1)^2(3)(2) = 18.8496\ ft^4$$

$$Ixy = 42.667 - 18.8496 = 23.817\ ft^4$$

Example



Solution

$$I_{xy1} = \frac{-(6)^2(3)^2}{72} + \frac{6*3}{2}(-4)(7) = -256.5\ in^4$$

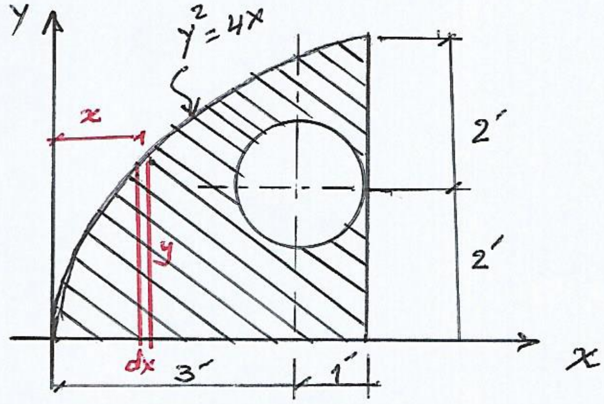
$$I_{xy2} = 0 + 6*6(-3)(3) = -324\ in^4$$

$$I_{xy3} = 0 + \pi(1)^2(-4)(-4) = -50.2655\ in^4$$

$$Ixy = -256.5 - 324 + 50.3655 = -530.235\ in^4$$

Example

Determine the product of inertia I_{xy} for the shaded area shown with respect to axes through the origin.



Solution

$$I_{xy1} = \int_0^4 x\frac{y}{2}ydx$$

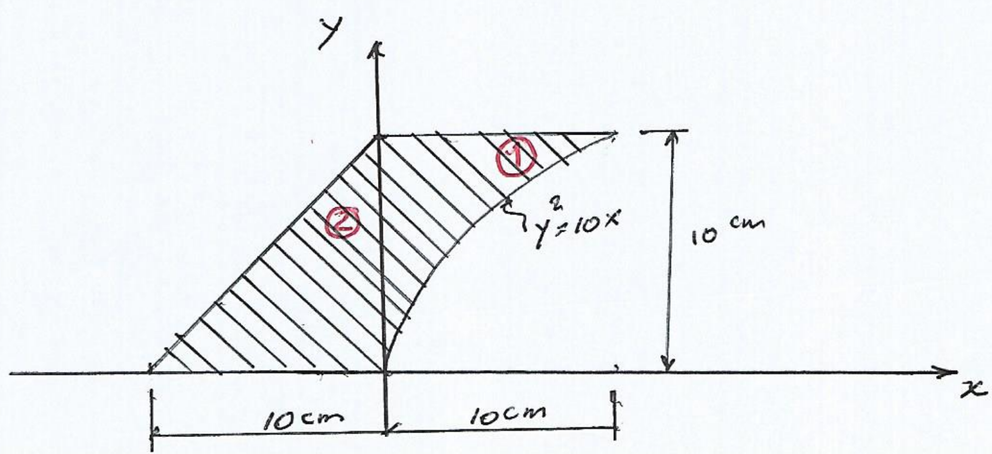
$$= \frac{1}{2}\int_0^4 4x^2 dx$$

$$= \frac{1}{2}[\frac{4x^3}{3}] = 42.667\ ft^4$$

$$I_{xy2} = 0 + \pi(1)^2(3)(2) = 18.8496\ ft^4$$

$$Ixy = 42.667 - 18.8496 = 23.817\ ft^4$$

Example



Solution

$$I_{xy1} = \int_0^{10} \frac{x}{2} yxdy$$

$$= \frac{1}{2}\int_0^{10} x^2 ydy = \frac{1}{2}\int_0^{10} \frac{y^4}{100} ydy$$

$$= \frac{1}{200}[\frac{y^6}{6}] = 833.3\ cm^4$$

$$I_{xy2} = \frac{(10)^2(10)^2}{72} + \frac{10(10)}{2}(\frac{-10}{3})(\frac{10}{3}) = -416.7\ cm^4$$

$$Ixy = 833.3 - 416.7 = 416.6\ cm^4$$

MOHR CIRCLE

Mohr circle is used with the moments of inertia for the following:

- to find moments of inertia with respect to rotated axes.
- To find the maximum and minimum moments of inertia and determine the angles of axes these moments work on.

To construct Mohr circle, follow the steps:

1- Find I_x , I_y and I_{xy}.

2- Draw the axes of Mohr circle where :

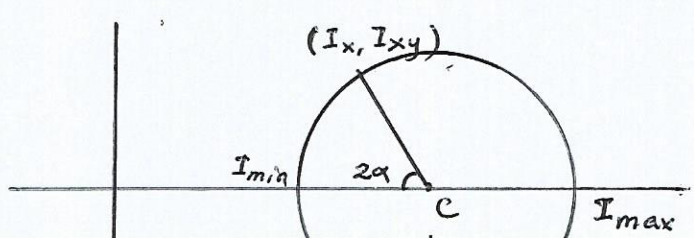
- X- Axis for rectangular moments of inertia
- Y- Axis for product moment of inertia I_{xy}

3- Locate the center of Mohr circle C, where : $C=\frac{I_x + I_y}{2}$ on X- axis.

4- Locate the control point A where $A(I_x, I_{xy})$.

5- Find the radius R of Mohr circle, where $R=\sqrt{(\frac{I_x + I_y}{2})^2 + I_{xy}^2}$ and draw Mohr circle.

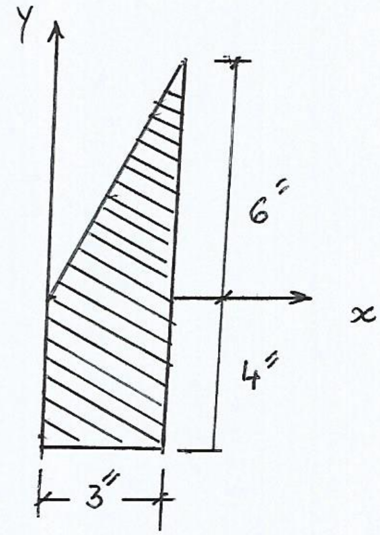
6- Find I_{max} and I_{min} where : $I_{max}=C + R$, $I_{min}=C - R$ and check that : $I_x+I_y=I_{min}+I_{max}$.



7- Find the angles of rotation for I_{max} and I_{min}

Example

Determine the maximum and minimum rectangular moments of inertia of the area shown with respect to axes through the origin.



Solution

$$I_x = \frac{3(4)^3}{3} + \frac{3(6)^3}{12} = 118\ in^4$$

$$I_y = \frac{4(3)^3}{3} + \frac{6(3)^3}{36} + \frac{3(6)}{2}(2)^2 = 76.5\ in^4$$

$$I_{xy} = 0 + 3*4*1.5(-2) + \frac{(3)^2(6)^2}{72} + \frac{3*6}{2}*2*2 = 4.5\ in^4$$

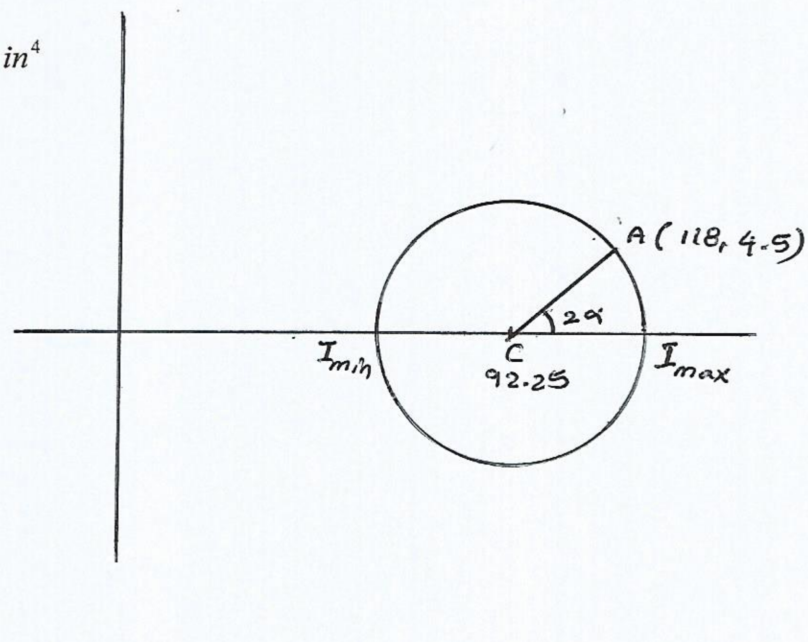
$$C = \frac{I_x + I_y}{2} = \frac{118 + 76.5}{2} = 97.25\ in^4$$

$$R = \sqrt{(\frac{118 - 76.5}{2})^2 + (4.5)^2} = 21.2323\ in^4$$

$Control\ Point\ A(118, 4.5)$

$I_{max} = 97.25 - 21.2323 = 118.5\ in^4$

$I_{min} = 97.25 - 21.2323 = 76.0\ in^4$



Check

$I_x + I_y = 118 + 76.5 = 194.5$

$I_{\min} + I_{\max} = 118.5 + 76 = 194.5 \quad OK$

$\sin(2\alpha) = \dfrac{4.5}{21.2323}$

$2\alpha = 12.236 \quad \alpha = 6.118^\circ$